

A Lagrangian form of Pfaff forms

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Abstract

The aim of the paper is to study some dynamic aspects coming from a Pfaff form, i.e. a time dependent differential form on a tangent bundle. The action on curves of a Pfaff form is natural associated with that of a second order Lagrangian linear in accelerations, while the converse association is not unique. An equivalence relation of Pfaff form, compatible with gauge equivalent Lagrangians, is considered. We express the Euler-Lagrange equation of the Lagrangian as a second order Lagrange derivative of a Pfaff form, considering controlled and higher order Pfaff forms. Hamiltonian forms of the dynamics generated are given, extending some quantization formulas given by Lukierski, Stichel and Zakrzewski. Using semi-sprays, local solutions of the E-L equations are given in some special particular cases.

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1 Introduction

The second order Lagrangians are considered, for example, in [5, 6], [15], [20] etc. The second order Lagrangians that are affine in acceleration are involved in some special problems and studied for example in [1], [3], [4], [5], [6], [9], [12], [13], [16] etc. These Lagrangians, are the most singular possible - their vertical hessian vanishes and according to [5, Sect. 6.3], some special regularity conditions can be considered. Third order Lagrangians that are affine in the third order derivatives, possessing an acceleration-extended Galilean symmetry, are studied in [10]; they extend the second order case considered previously by the authors and considered in a general form in this paper. It can be a model for a future development of constructions in the present paper.

In this paper we study Pfaff forms, i.e. differentiable one forms ω on $\mathbb{R} \times TM$, where M is a manifold. Some basic aspects and motivating examples can be found in our previous paper [18]. We consider an action of a Pfaff form ω on differentiable curves on M , in fact the same as the action of a second order

Lagrangian affine in accelerations that corresponds canonically to ω (Proposition 2.1). Conversely, the action of a second order Lagrangian affine in accelerations can correspond to at least one Pfaff form (Proposition 2.2). We consider a certain equivalence relation on Pfaff forms such that a such equivalence class corresponds to gauge equivalent Lagrangians that give the actions (Proposition 3.1).

Considering controlled Pfaff forms (Proposition 3.2), higher order Pfaff forms, top Pfaff forms and Lagrange derivatives of Pfaff forms, then the Euler Lagrange equation of a Pfaff form can be obtained by (two) successive Lagrange derivatives of Pfaff forms (Proposition 4.1), considering an Ostrogradski Pfaff form, closed related to Ostrogradski momenta. The Euler-Lagrange equation contains the second derivatives and we prove that in the case of a regular Lagrangian, the solutions are integral curves of a global second order differential equation (Proposition 4.2).

Considering a Legendre map and defining non-degenerated, hyper-non-degenerated and biregular Lagrangians, we study the dynamics given by Lagrangians linear in accelerations, defined using Pfaff forms. We prove that for a regular Pfaff form, the dynamics on M (i.e. the solutions of E-L equations) comes from the projection of the integral curves of a vector field X on $T^{2*}M = T^*M \times_M TM$ (Proposition 5.1), while for a biregular Pfaff form, the dynamics on M comes from the projection of the integral curves of a vector field Y on $T_2^0M = TM \times_M TM$ (Proposition 5.2).

Important tools in describing the dynamic equations of a Hamiltonian system are offered by quantizations. Following similar ideas used in [9, Section 2.], where Ostrogradski-Dirac and Fadeev-Jakiw methods are used, we use here a modified Ostrogradski-Dirac method, offered by the possibility to construct constraints slight different from the canonical ones used in Ostrogradski theory. The Ostrogradski-Dirac method was also used in [3] in the quantization of the system derived from a Lagrangian linear in velocities, involved in the study of a Reegge-Teitelboim model. Since in the cases considered in our paper it is not necessary to express the constraints technics explicitly, we use an symplectic formalism instead, giving here a global form of the quoted methods. In Subsection 5.2 we present a Hamiltonian description of the dynamics defined by the vector fields X and Y described above, proving that:

- if ω is regular and its essential part is time independent, then there are symplectic forms Ξ'_t on $T^{2*}M$, $t \in \mathbb{R}$, and a hamiltonian $H : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field X_H is X (Theorem 5.1);
- if ω is biregular and its essential part is time independent, then there are symplectic forms Ξ''_t on T_2^0M , $t \in \mathbb{R}$, and a hamiltonian $H' : \mathbb{R} \times T_2^0M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field $X_{H'}$ is Y (Theorem 5.2).

Some examples and special cases are given in Subsection 6. In the case when $\dim M = 1$, we prove in Proposition 6.2 that the generalized Euler-Lagrange equation of a regular and basic Pfaff form admits locally standard Lagrangian descriptions (in the sense of [2, Section 2.]). In order to describe the dynamics generated by some classes of Pfaff forms, we use first order semi-sprays. Following some concrete examples, we consider some special cases (Propositions 6.3 to

6.7) when families of local semi-sprays of first order are considered, such that their integral curves project on (sometimes all) integral curves of the generalized Euler-Lagrange equation associated with the Lagrangian of the Pfaff form.

Using local calculus, certain geometrical objects on higher order tangent bundles and on general fibered manifolds are described in an Appendix.

2 Pfaff forms, Lagrangians and actions on curves

A *Pfaff form* is a differentiable form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$. The global generator $dt \in \mathcal{X}^*(\mathbb{R})$ gives the global pull-back form $p_1^*dt \in \mathcal{X}^*(\mathbb{R} \times TM)$, denoted also by dt , where we denote by $p_1 : \mathbb{R} \times TM \rightarrow \mathbb{R}$ and $p_2 : \mathbb{R} \times TM \rightarrow TM$ the natural projections. The local generators $\{dx^i, dy^i\}$ of local forms on TM , on a domain of local coordinates $TU = U \times \mathbb{R}^m$, give the local generators $p_2^*dx^i \in \mathcal{X}^*(\mathbb{R} \times TU)$ and $p_2^*dy^i \in \mathcal{X}^*(\mathbb{R} \times TU)$, denoted also by dx^i and dy^i respectively. We obtain the local $\mathcal{F}(\mathbb{R} \times TM)$ -module bases $\{dt, dx^i, dy^i\}$ of Pfaff forms. Using these local bases, a Pfaff form ω has the local form

$$\omega = \omega_0(t, x^i, y^i)dt + (\omega_i(t, x^j, y^j)dx^i + \bar{\omega}_i(t, x^j, y^j)dy^i) = \omega_0 + \omega', \quad (1)$$

where

$\omega' = \omega_i(t, x^j, y^j)dx^i + \bar{\omega}_i(t, x^j, y^j)dy^i$ is a new (global) Pfaff form that we call the *essential component* of ω ;

$\omega_0 : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a (global) Lagrangian that we call the *Lagrangian component* of ω .

We say that the Pfaff form $\omega = \omega_0 + \omega'$ is *pure* if $\omega_0 = 0$, *Lagrangian* if $\omega' = 0$ and *mixed* if ω is neither pure, nor Lagrangian.

It is easy to see that a first order Lagrangian $L : TM \rightarrow \mathbb{R}$ is the same as the Lagrangian Pfaff form $\omega_0 = Ldt$.

Let $\eta : \mathbb{R} \times TM \rightarrow \pi^*T^*M$, having local the form $\eta = \eta_i(t, x^j, y^j)dx^i$ and coming from a section of the induced vector bundle $\mathbb{R} \times p_2^*T^*M \rightarrow \mathbb{R} \times TM$. It can be regarded as well as a Pfaff form, and is called a *top Pfaff form*. A Pfaff form ω having the form (1) defines a top Pfaff form $\tilde{\omega} = \bar{\omega}_i(t, x^i, y^i)dx^i$ (see Appendix). According to a definition given below, the top Pfaff form $\tilde{\omega}$ is degenerated when it is regarded as a Pfaff form.

A Pfaff form can be related to a second order dynamic form considered in [5]. According to [5, Section 2], a *first order dynamic form* on the bundle $Y = \mathbb{R} \times M \rightarrow M$ is a one contact and horizontal two form ν on $J^1(Y)$, having the local form $\nu = \nu_i(t, x^j, y^j)dx^i \wedge dt + \bar{\nu}_i(t, x^j, y^j)dy^i \wedge dt$. Obviously a first order dynamic form is equivalent to give a pure Pfaff form. An advantage to use Pfaff forms is having the Lagrangian forms in the same setting. An other motivation to use Pfaff forms is given by their action on curves, that relates them to the second order Lagrangians that are affine in accelerations, i.e. the vertical Hessian vanishes.

If $\gamma : [a, b] \rightarrow M$ is a curve on M , then for every $t \in [a, b]$ one can consider in $\tilde{\gamma}(t) = (t, \frac{d\gamma}{dt}(t)) \in \mathbb{R} \times TM$ the scalar $\omega_{\tilde{\gamma}(t)}\left(\frac{d^2\gamma}{dt^2}(t)\right)$. The *action* of the Pfaff form ω on γ is given by the formula

$$I_\omega(\gamma) = \int_a^b \omega_{\tilde{\gamma}(t)}\left(\frac{d^2\gamma}{dt^2}(t)\right) dt. \quad (2)$$

Let us relate the action of Pfaff forms on curves to the actions of Lagrangians on curves. First, we define the *action* of a first order Lagrangian $L^{(1)} : \mathbb{R} \times TM \rightarrow \mathbb{R}$ on a curve $\gamma : [a, b] \rightarrow M$, by the formula:

$$I_L(\gamma) = \int_a^b L^{(1)}\left(t, \gamma(t), \frac{d\gamma}{dt}(t)\right) dt.$$

If $\gamma : [a, b] \rightarrow M$ is a curve on M , then the curves $\frac{d\gamma}{dt} : [a, b] \rightarrow TM$ (the *velocity curve*) and $\frac{d^2\gamma}{dt^2} : [a, b] \rightarrow T^2M \subset TTM$ (the *acceleration curve*) are called here the *first order lift* and the *second order lift* respectively, of the curve γ . A *second order Lagrangian* on M is a differentiable map $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$, where T^2M is the second order tangent space of M (see Appendix). Then $L^{(2)}$ defines also an *action* on γ by formula:

$$I_\omega(\gamma) = \int_a^b L^{(2)}\left(t, \gamma(t), \frac{d\gamma}{dt}(t), \frac{d^2\gamma}{dt^2}(t)\right) dt.$$

A second order Lagrangian L is *affine in accelerations* if its vertical Hessian vanishes. Using local coordinates, $L(t, x^i, y^i, z^i) = f_0(t, x^i, y^i) + z^i g_i(t, x^i, y^i)$. Notice that if $g_i = 0$, then $L = f_0$ is a first order Lagrangian. In this case f_0 is obtained projecting $L : T^2M \rightarrow \mathbb{R}$ on $f_0 : TM \rightarrow \mathbb{R}$, by the natural projection $T^2M \rightarrow TM$. Considering the degeneration of this situation is refined below in the paper.

It is easy to see that the Lagrangian action I_{L_0} of a Lagrangian L_0 is the same as the Pfaff action I_{ω_0} of the Lagrangian Pfaff form $\omega_0 = L_0 dt \in \mathcal{X}^*(\mathbb{R} \times TM)$. It worth to remark that ω_0 is a closed form only if $L_0 = L_0(t)$. Let us remark also that the formula (2) is free of coordinates and gives an easy tool to obtain an action involving accelerations, velocities, position and time. More precisely, the accelerations are involved affinely, as follows.

Proposition 2.1 *If $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ is a Pfaff form, then there is a second order Lagrangian $L^{(2)} : T^2M \rightarrow \mathbb{R}$, affine in accelerations, such that $I_\omega = I_L$.*

Proof. Using local coordinates: $t \in \mathbb{R}$, (x^i) on M and (x^i, y^i) on TM , then ω has the local form $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$, where the local functions ω_0 , (ω_i) and $(\bar{\omega}_i)$ locally depend on (t, x^i, y^i) . If the curve γ has the local form $t \rightarrow (x^i(t))$, then the action (2) has the local form

$$I_\omega(\gamma) = \int_a^b (\omega_0 + \omega_i \frac{dx^i}{dt} + \bar{\omega}_i \frac{d^2x^i}{dt^2}) dt. \quad (3)$$

Using coordinates (x^i, y^i, z^i) on T^2M , we define the Lagrangian

$$L_\omega^{(2)} : T^2M \rightarrow \mathbb{R}, \quad L^{(2)}(t, x^i, y^i, z^i) = \omega_0(t, x^i, y^i) + \omega_i(t, x^j, y^j)y^i + \bar{\omega}_i(t, x^j, y^j)z^i. \quad (4)$$

It is easy to see that the action of $L_\omega^{(2)}$ on a curve γ is given also by formula (3), thus the conclusion follows. \square

The following result shows that the action of every second order Lagrangian, affine in accelerations, can be represented as well as an action of a suitable Pfaff form.

Proposition 2.2 *Let $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$ be a second order Lagrangian affine in accelerations. Then there is a Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ such that $I_\omega = I_L$.*

Proof. Let us consider a local chart (U, φ) on M ; we define a locally Pfaff form $\theta_U = \frac{\partial L}{\partial z^i} dy^i$, thus $(\theta_U)_i = \frac{\partial L}{\partial z^i}$ and $(\theta_U)_i = (\theta_U)_0 = 0$ for this Pfaff form. Let $\{f_\alpha\}_{\alpha \in N}$ be a partition of unity subordinated to an open cover $\{U_\alpha\}_{\alpha \in N}$ of domains of coordinates, that is locally finite. Then the Pfaff form $\theta = \sum_{\alpha \in N} f_\alpha \cdot \theta_{U_\alpha}$ is a Pfaff form $\theta \in \mathcal{X}^*(\mathbb{R} \times TM)$ that has

the top component $\bar{\theta}_i = (\bar{\theta}_U)_i = \frac{\partial L}{\partial z^i}$ and $\theta = \bar{\theta}_i dy^i + \theta_i dx^i$. Since $L^{(2)}$ has the local form $L^{(2)}(t, x^i, y^i, z^i) = \bar{\theta}_i(t, x^j, y^j)z^i + u(t, x^j, y^j)$, one has also $L^{(2)}(t, x^i, y^i, z^i) = \bar{\theta}_i(t, x^i, y^i)z^i + \theta_i(t, x^i, y^i)y^i + (u(t, x^i, y^i) - \theta_i y^i)$. The local functions $L_0(t, x^i, y^i) = u(t, x^i, y^i) - \theta_i y^i$ give a global function $L_0 : \mathbb{R} \times TM \rightarrow \mathbb{R}$. Thus the Pfaff form $\omega = \theta + L_0 dt$ has the property that $I_\omega = I_L$. \square

The actions of Pfaff forms on curves are related to the well-known actions of the first and the second order Lagrangians on curves. Let us consider two points $x, y \in M$ and $\gamma_0 = (x_0^i(t))$ be a curve joining x and y , i.e. $x_0^i(0) = x$ and $x_0^i(1) = y$. Let us consider variations of γ_0 , as curves joining x and y , having the local form $\gamma_\varepsilon = (x_\varepsilon^i(t))$, where $x_\varepsilon^i(t) = x_0^i(t) + \varepsilon h^i(t)$.

In the case of the actions of second order Lagrangians on curves, the specific variational conditions, impose:

$$h^i(a) = h^i(b) = 0, \quad (5)$$

$$\frac{dh^i}{dt}(a) = \frac{dh^i}{dt}(b) = 0. \quad (6)$$

For a second order Lagrangian $L^{(2)} : \mathbb{R} \times T^2M \rightarrow \mathbb{R}$, the extrema curves of the action $I_{L^{(2)}}$ are given by the well-known Euler-Lagrange equations

$$\frac{\partial L^{(2)}}{\partial x^i} - \frac{d}{dt} \frac{\partial L^{(2)}}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L^{(2)}}{\partial z^i} = 0. \quad (7)$$

In the particular case of a Lagrangian (4), the Euler-Lagrange equations have the form

$$\frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} \frac{dx_0^j}{dt} + \frac{\partial \bar{\omega}_j}{\partial x^i} \frac{d^2 x_0^j}{dt^2} - \frac{d}{dt} \left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} \frac{dx_0^j}{dt} + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i} \frac{d^2 x_0^j}{dt^2} \right) + \frac{d^2}{dt^2} \bar{\omega}_i = 0. \quad (8)$$

Let us consider \mathbb{R}^2 with coordinates x and y . The canonical symplectic form $\alpha = dx \wedge dy$ gives the Pfaff form $\omega^{(1)} = xdy - ydx$ and the second order Lagrangian $L_0(t, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = \dot{x}\ddot{y} - \dot{y}\ddot{x}$ on \mathbb{R}^2 ; here $(x, y) := (x^1, x^2)$; $(\dot{x}, \dot{y}) := (y^1, y^2)$, in the previous notations. This Lagrangian was involved in [9], concerning its invariance to the $(2 + 1)$ -Galilean symmetry; the authors prove in the Appendix that *the general form of a one-particle Lagrangian which is at most linearly dependent on \ddot{x} and \ddot{y} leading to Euler-Lagrange equations of motion which are covariant with respect to the $D = 2$ Galilei group, is given, up to gauge transformations, by $L(t, x, y, \dot{x}, \dot{y}, \ddot{x}, \ddot{y}) = -k(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. This Lagrangian is affine in velocities, but it can come from two Pfaff forms:*

$$\begin{aligned}\omega_1 &= -k(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{m}{2}(\dot{x}dx + \dot{y}dy), \\ \omega_2 &= -k(\dot{x}\ddot{y} - \dot{y}\ddot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)dt.\end{aligned}\tag{9}$$

In order to put together these two Pfaff forms, we define below an equivalence relation, ruled by their action and implicitly by their second order Lagrangians, affine in velocities.

3 Equivalence of Pfaff forms

A first order Lagrangian $F : \mathbb{R} \times TM \rightarrow \mathbb{R}$ and a Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ gives the Pfaff form $\omega' = \omega + dF$. Then

$$I_{\omega'}(\gamma) = I_{\omega}(\gamma) + \int_a^b \left(\frac{\partial F}{\partial t} + \frac{\partial F}{\partial x^i} \frac{dx^i}{dt} + \frac{\partial F}{\partial y^i} \frac{dy^i}{dt} \right) dt = I_{\omega}(\gamma) + F(x^i(b), \frac{dx^i}{dt}(b)) - F(x^i(a), \frac{dx^i}{dt}(a)).$$

According to the variation conditions (5) and (6), it is easy to see that I_{ω} and $I_{\omega'}$ have the same extrema curves.

Analogous considerations as made in [8] for the gauge equivalence of first order Lagrangians can be transposed for second order Lagrangians (see for example [14, Section 4.4]). It reads that the second order Lagrangians L and $L' = L + \frac{d}{dt}F$, $F : \mathbb{R} \times TM \rightarrow \mathbb{R}$, are gauge equivalent, i.e. they have the same extrema curves. Here $\frac{d}{dt}F$ stands for L_{dF} , the second order Lagrangian associated with the Pfaff form dF . The analogous gauge form for actions of the corresponding Pfaff forms, reads that the Pfaff forms ω and $\omega' = \omega + dF$ have the same extrema curves.

We notice that the Lagrangians given by [4, formula (11)] or [3, formula (34)] are gauge equivalent, but they are studied without using this fact.

Let us consider the submodule $\mathcal{G} \subset \mathcal{X}^*(\mathbb{R} \times TM)$ generated by the local differential forms $\{\delta x^i = dx^i - y^i dt\}_{i=1, \dots, m}$. A form $\eta \in \mathcal{G}$ iff it has the local form $\eta = a_i(t, x^j, y^j) \delta x^i$. It is easy to see that any form $\eta \in \mathcal{G}$ vanishes along the (second order) lift of a curve on M . Thus for any Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$, the Pfaff forms ω and $\omega' = \omega + \eta$ have the same extrema curves (see [13] for other implications concerning the module \mathcal{G}).

We say that:

two Pfaff forms $\omega, \omega' \in \mathcal{X}^*(\mathbb{R} \times M)$ are *equivalent* if there is an $F \in \mathcal{X}^*(\mathbb{R} \times M)$ such that $\omega' - \omega - dF \in \mathcal{G}$;

two second order Lagrangians L' and L are *gauge equivalent* if there is an $F \in \mathcal{X}^*(\mathbb{R} \times TM)$ such that $L' - L = \frac{d}{dt}F$.

It is easy to see that two equivalent Pfaff forms have the same extrema curves. Analogously, two second order Lagrangians L' and L that are gauge equivalents have the same extrema curves.

Proposition 3.1 *Two Pfaff forms ω' and ω are equivalent iff their second order Lagrangians $L_{\omega'}$ and L_{ω} are gauge equivalent.*

Proof. Let $\omega' = \bar{\omega}'_i dy^i + \omega'_i dx^i + \omega'_0$ and $\omega = \bar{\omega}_i dy^i + \omega_i dx^i + \omega_0$ be equivalent. Thus there is $F \in \mathcal{F}(\mathbb{R} \times M)$ such that $\omega' - \omega - dF = \eta_i(dx^i - y^i dt)$. It is easy to see that $L_{\omega'} - L_{\omega} = \frac{d}{dt}F$, thus $L_{\omega'}$ and L_{ω} are gauge equivalent. Conversely, Let us suppose that $L_{\omega'}$ and L_{ω} are gauge equivalent, thus $L_{\omega'} - L_{\omega} = \frac{d}{dt}F$. Then $\bar{\omega}'_i = \bar{\omega}_i + \frac{\partial F}{\partial y^i}$ and $\omega'_i y^i + \omega'_0 = (\frac{\partial F}{\partial y^i} + \omega_i)y^i + \frac{\partial F}{\partial t} + \omega_0$. It follows that $\omega' - \omega - dF = (\omega'_i - \omega_i - \frac{\partial F}{\partial x^i})(dx^i - y^i dt) \in \mathcal{G}$, thus ω' and ω are equivalent. \square

It follows that the property of the above Proposition can be used as a definition of equivalent Pfaff forms.

Corrolary 3.1 *If two Pfaff forms correspond to the same second order Lagrangian affine in accelerations, then they are equivalent.*

The Poincaré-Cartan form $\theta_L = Ldt + \frac{\partial L}{\partial y^i} \delta x^i$ of a first order Lagrangian L is obviously equivalent to the canonical Lagrangian form Ldt and both correspond to the same Lagrangian L , seen of second order by $T^2M \rightarrow TM \rightarrow \mathbb{R}$.

There are two possibilities to associate a Pfaff form to a pointed Lagrangian $L(t, x^i, y^i) = y^i \nu_i$: $\omega_1 = \nu_i dx^i$ and $\omega_2 = Ldt$ respectively. The first is pure and the second is a Lagrangian one, but they have the same action on curves, that given by the action of the same Lagrangian. It is easy to see that $d\omega_1 - d\omega_2 = 0$ iff $\nu^i = 0$; thus ω_1 and ω_2 are not differential equivalent (i.e. $\omega_1 - \omega_2 = dF$) for $L \neq 0$. Thus there are Pfaff forms that are not differential equivalent (i.e. their difference is not a exact differential), but equivalent.

Every Pfaff form ω that has the local form $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$ is locally equivalent to the local Pfaff form $\omega' = (\omega_0 + y^i \omega_i)dt + \bar{\omega}_i dy^i$, since $\omega - \omega' = \omega_i \delta x^i$. But, in general ω' is not a global Pfaff form.

The Pfaff forms $\omega_1 = -k(\dot{x}dy - yd\dot{x}) + \frac{m}{2}(\dot{x}dx + \dot{y}dy)$ and $\omega_2 = -k(\dot{x}dy - yd\dot{x}) + \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$ considered previously are equivalent, since $\omega_1 - \omega_2 = \frac{m}{2}(\dot{x}(dx - \dot{x}dt) + \dot{y}(dy - \dot{y}dt))$. Let us consider below two other situations.

1) Considering the canonical symplectic form in \mathbb{R}^2 given by $(\varepsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

we obtain the pure Pfaff form $\omega_1 = -k\varepsilon_{ij}y^i dy^j$, or $\omega_1 = -k\varepsilon_{ij}\dot{x}^i \dot{x}^j$ where k is a non-null constant, that corresponds to the second order Lagrangian $L_0(x^i, y^i, z^i) = -k\varepsilon_{ij}y^i z^j$, or $L_0(x^i, \dot{x}^i, \ddot{x}^i) = -k\varepsilon_{ij}\dot{x}^i \ddot{x}^j$.

The Lagrangian $L(x^i, y^i, z^i) = \frac{my^i y^j \delta_{ij}}{2} - k\varepsilon_{ij} y^i z^j$, or $L(x^i, \dot{x}^i, \ddot{x}^j) = \frac{m\dot{x}^i \ddot{x}^j \delta_{ij}}{2} - k\varepsilon_{ij} \dot{x}^i \ddot{x}^j$ was considered in [9, 10, 1].

In order to obtain a Pfaff form we have two possibilities: $\omega = my^i \delta_{ij} dx^j - k\varepsilon_{ij} y^i dy^j$ and $\omega' = \frac{my^i y^j \delta_{ij}}{2} dt - k\varepsilon_{ij} y^i dy^j$; the first is pure and the second is a mixed one.

2) The Lagrangian $L(x^i, y^i, z^i) = -m \|y^{(1)}\| + \frac{\varepsilon_{ij} y^i z^j}{\|y^{(1)}\|^3}$, where $\|y^{(1)}\| = \sqrt{\frac{y^i y^j \delta_{ij}}{2}}$ or $L(x^i, \dot{x}^i, \ddot{x}^j) = -m \|\dot{x}\| + \frac{\varepsilon_{ij} \dot{x}^i \ddot{x}^j}{\|\dot{x}\|^3}$, where $\|\dot{x}\| = \sqrt{\frac{m\dot{x}^i \ddot{x}^j \delta_{ij}}{2}}$ was considered in [12]. The two Pfaff forms, one pure and one mixed, can also be considered: $\omega = -\frac{my^i \delta_{ij}}{\|y^{(1)}\|} dx^j + \frac{\varepsilon_{ij} y^i}{\|y^{(1)}\|^3} dy^j$, and $\omega' = -m \|y^{(1)}\| dt + \frac{\varepsilon_{ij} y^i}{\|y^{(1)}\|^3} dy^j$.

Unlike the first example, in the second example the Pfaff form ω' is not differentiable in the points where $(y^i = 0)$.

3.1 Controlled and higher order Pfaff forms

Let us extend the top Lagrange derivative to a broader class of Pfaff forms.

Let $\pi_E : E \rightarrow TM$ a weak fiber manifold over the tangent bundle (i.e. a submersion) such that the composed projection $\pi_M : E \rightarrow M$ is a fiber manifold (i.e. a surjective submersion). A *bundle map* of two fibered manifolds over the base M is a map that send fibers to fibers ($E_x = \pi_M^{-1}(x)$ is the fiber of $x \in M$). We denote by $\pi_{TM} : TM \rightarrow M$ the canonical projection. Using local coordinates (x^i) on M , (x^i, u^α) on E and (x^i, y^j) on TM that are adapted to the fibered manifold structures, then we have the local forms $(x^i, y^j) \xrightarrow{\pi_{TM}} (x^i)$, $(x^i, u^\alpha) \xrightarrow{\pi_E} (x^i)$, $(x^i, y^j, u^\alpha) \xrightarrow{\pi_E} (x^i, y^j)$, $(x^i, u^\alpha, y^j, v^\beta) \xrightarrow{\pi_E} (x^i, u^\alpha)$ and $(x^i, y^j, u^\alpha, v^\beta) \xrightarrow{\pi_E} (\rho^i(x^i, y^j, u^\alpha, v^\beta))$. If coordinates change, then one follow the rules: $x^{i'} = x^i(x^i)$, $y^{j'} = \frac{\partial x^{i'}}{\partial x^i} y^j$, $u^{\alpha'} = u^\alpha(x^i, u^\alpha)$, $v^{\beta'} = \frac{\partial u^{\beta'}}{\partial x^i} y^i + \frac{\partial u^{\beta'}}{\partial u^\alpha} v^\alpha$.

A *controlled Pfaff form* on E is bundle map $\omega : \mathbb{R} \times E \rightarrow T^*TM$ over the base M . A *controlled top Pfaff form* on E is a bundle map $\bar{\omega} : \mathbb{R} \times E \rightarrow T^*M$ over the base M .

Using local coordinates a controlled Pfaff form ω has the local form, $\omega = \omega_i(t, x^i, u^\alpha) dx^i + \bar{\omega}_i(t, x^i, u^\alpha) dy^i$; the local functions change according to the rules $\bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}$ and $\omega_i = \frac{\partial y^{j'}}{\partial x^i} \bar{\omega}_{j'} + \frac{\partial x^{i'}}{\partial x^i} \omega_{i'}$ respectively. Analogously, a controlled top Pfaff form $\bar{\omega}$ has the form $\bar{\omega} = \bar{\omega}_i dx^i$ and $\bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}$. Obviously a controlled Pfaff form ω as above give rise to a top Pfaff form $\bar{\omega}$.

Let us define the Lagrange controlled *top derivative* of ω as

$$\mathcal{E}'_\omega(t, x^i, y^j, u^\alpha, v^\alpha) = [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \frac{\partial \bar{\omega}_i}{\partial u^\alpha} v^\alpha + \frac{\partial \bar{\omega}_i}{\partial t})] dx^i. \quad (10)$$

Proposition 3.2 *The Lagrange controlled top derivative of ω is a global map $\mathcal{E}'_\omega : \mathbb{R} \times TE \rightarrow T^*M$, over M .*

Proof. We have $\omega_i - (\frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \frac{\partial \bar{\omega}_i}{\partial u^\alpha} v^\alpha + \frac{\partial \bar{\omega}_i}{\partial t}) = \frac{\partial y^{j'}}{\partial x^i} \bar{\omega}_{j'} + \frac{\partial x^{i'}}{\partial x^i} \omega_{i'} - y^j \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} \bar{\omega}_{i'} - y^j \frac{\partial x^{i'}}{\partial x^i} (\frac{\partial u^{\alpha'}}{\partial x^j} \frac{\partial \bar{\omega}_{i'}}{\partial u^{\alpha'}} + \frac{\partial x^{j'}}{\partial x^j} \frac{\partial \bar{\omega}_{i'}}{\partial x^{j'}}) - \frac{\partial x^{i'}}{\partial x^i} \frac{\partial \bar{\omega}_{i'}}{\partial u^{\alpha'}} \frac{\partial u^{\alpha'}}{\partial x^i} v^\alpha - \frac{\partial x^{i'}}{\partial x^i} \frac{\partial \bar{\omega}_{i'}}{\partial t} =$

$$\frac{\partial x^{i'}}{\partial x^i}(\omega_{i'} - \frac{\partial \bar{\omega}_{i'}}{\partial x^{j'}} y^{j'} - \frac{\partial \bar{\omega}_{i'}}{\partial u^\alpha} v^{\alpha'} - \frac{\partial \bar{\omega}_{i'}}{\partial t}). \quad \square$$

Considering the (local) operator $\frac{d}{dt} = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial u^\alpha}$, then $\mathcal{E}'_\omega = (\omega_i - \frac{d}{dt} \bar{\omega}_i) dx^i$.

If $E = TM$, we obtain the definition of a controlled Pfaff form ω or top Pfaff form, as above. The Lagrange controlled derivative of $\omega : \mathcal{E}'_\omega : \mathbb{R} \times TTM \rightarrow T^*M$ restricts to the Lagrange derivative of $\omega : \mathcal{E}'_\omega : \mathbb{R} \times T^2M \rightarrow T^*M$ given above by formula (10), since $T^2M \subset TTM$; using local coordinates, this inclusion looks as $(x^i, y^i, z^i) \rightarrow (x^i, y^i, y^i, z^i)$.

We call:

a *k-order Pfaff form* any controlled Pfaff form $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$ and

a *k-order top Pfaff form* any bundle map $\bar{\omega} : \mathbb{R} \times T^kM \rightarrow T^*M$.

It is easy to see that any *k-order Pfaff form* $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ gives a *k-order top Pfaff form* $\bar{\omega} = \bar{\omega}_i dx^i$.

Let us define now the *Lagrange top derivative* $\mathcal{E}_\omega^{(k+1)}$ of a *k-order Pfaff form* $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$. The Lagrange controlled top derivative of ω is $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M$, according to the inclusion $T^{k+1}M \subset T^{k+1}M$. Using local coordinates, the inclusion has the form $(x^i, y^i, \dots, w^i, \tilde{w}^i) \rightarrow (x^i, y^i, \dots, w^i, y^i, \dots, w^i, \tilde{w}^i)$, $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M$ has the local form

$$\mathcal{E}'_\omega(t, x^i, y^i, \dots, w^i, X^i, Y^i, \dots, W^i) = [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial t} + \frac{\partial \bar{\omega}_i}{\partial x^j} X^j + \frac{\partial \bar{\omega}_i}{\partial y^j} Y^j + \dots + \frac{\partial \bar{\omega}_i}{\partial w^j} W^j)] dx^i$$

and the restriction to $\mathbb{R} \times T^{k+1}M$ is

$$\begin{aligned} \mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1}M \rightarrow T^*M, \mathcal{E}_\omega^{(k+1)}(t, x^i, y^i, \dots, z^i, w^i, \tilde{w}^i) = \\ [\omega_i - (\frac{\partial \bar{\omega}_i}{\partial t} + \frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \dots + \frac{\partial \bar{\omega}_i}{\partial z^i} w^i + \frac{\partial \bar{\omega}_i}{\partial w^i} \tilde{w}^i)] dx^i \end{aligned}$$

, or

$$\mathcal{E}_\omega^{(k+1)} = [\omega_i - \frac{d}{dt} \bar{\omega}_i] dx^i, \quad (11)$$

where $\frac{d}{dt}$ is the local operator given by $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + \dots + w^i \frac{\partial}{\partial z^i} + \tilde{w}^i \frac{\partial}{\partial w^i}$.

The first order Lagrange top derivative of a Pfaff form $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ is the second order top Pfaff form $\mathcal{E}_\omega^{(2)} = \mathcal{E}'_\omega : \mathbb{R} \times T^2M \rightarrow T^*M$ given using (11).

In the case $k = 3$, the local operator $\frac{d}{dt}$ is given by $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^i \frac{\partial}{\partial y^i} + w^i \frac{\partial}{\partial z^i} + \tilde{w}^i \frac{\partial}{\partial w^i}$. In the case when $\bar{\omega} = \bar{\omega}_i dx^i$ has the order 2, then $\frac{\partial \bar{\omega}_i}{\partial w^i} = 0$ and the third order Lagrange top derivative $\mathcal{E}_\omega^{(3)}$ has the order at most 3, as ω . This is the case below when the Euler-Lagrange top form of a Pfaff form has the third order.

We say that a *k-order Pfaff form* $\omega : \mathbb{R} \times T^kM \rightarrow T^*TM$ is *effectively of order k* (or the order *k* is *effective*) if ω can not be induced by a $(k-1)$ -Pfaff form $\omega' : \mathbb{R} \times T^{k-1}M \rightarrow T^*TM$ (using the canonical projection $T^kM \rightarrow T^{k-1}M$).

Analogously, a k -order top Pfaff form $\omega : \mathbb{R} \times T^k M \rightarrow T^* M$ is *effectively of order k* if ω can not be induced by a $(k-1)$ -Pfaff form $\bar{\omega}' : \mathbb{R} \times T^{k-1} M \rightarrow T^* M$.

The Lagrange top derivative of an effective k -order Pfaff form $\omega : \mathbb{R} \times T^k M \rightarrow T^* M$ is at most effective $(k+1)$ -order top Pfaff form $\mathcal{E}_\omega^{(k+1)} : \mathbb{R} \times T^{k+1} M \rightarrow T^* M$. In the case when an effective k -order Pfaff form $\omega = \omega_i dx^i + \bar{\omega}_i dy^i$ has the associated top Pfaff form $\bar{\omega} = \bar{\omega}_i dx^i$ of an effective s -order, $s \leq k-1$, then the Lagrange top derivative $\mathcal{E}_\omega^{(k+1)}$ has an effective k -order, as ω .

It is easy to see that if $\bar{\omega} : \mathbb{R} \times T^k M \rightarrow T^* M$, $\bar{\omega}(t, x^i, y^i, \dots, z^i) = \bar{\omega}_i(t, x^i, y^i, \dots, z^i) dx^i$ is a k -order top Pfaff form, then $(E_{ij} = \frac{\partial \bar{\omega}_i}{\partial z^j})$ and $(E_{ijk} = \frac{\partial^2 \bar{\omega}_i}{\partial z^j \partial z^k})$ are covariant tensors: a bilinear form E'_ω and a trilinear form E''_ω respectively, on the fibers of $T^* M$. Then $\bar{\omega}$ is effective of order k iff $E'_\omega \neq 0$. We say the k -order top Pfaff form $\bar{\omega}$ is *affine in k -accelerations* if $E''_\omega = 0$.

We say also that a k -order Pfaff form ω , with $\bar{\omega}$ the associated Pfaff form, is *affine in k -accelerations* if its Lagrange top derivative $\mathcal{E}_\omega^{(k+1)}$ is affine in k -accelerations (if the effective order of $\bar{\omega}$ is less than the effective order of ω) or in $(k+1)$ -accelerations (if the effective orders of ω and $\bar{\omega}$ are the same).

Using relation (11) it is easy to see that if a k -order Pfaff form ω and its associated top Pfaff form $\bar{\omega}$ have effectively the orders k , then the Lagrange top derivative of ω is affine in the $(k+1)$ -accelerations, i.e. denoting $\bar{\theta} = \mathcal{E}_\omega^{(k+1)}$, then $E''_{\bar{\theta}} = 0$ and $E'_{\bar{\theta}} = \mathcal{E}_{\bar{\omega}}^{(k+1)}$, specifically $E_{ij} = \frac{\partial \bar{\omega}_i}{\partial z^j}$.

4 The Euler-Lagrange equation as a top Pfaff form

Any second order Lagrangian $L : \mathbb{R} \times T^2 M \rightarrow \mathbb{R}$ gives rise to at most forth order top Pfaff form $\mathcal{E}_i dx^i$, that we call the *Euler-Lagrange top Pfaff form* of L , where

$$\mathcal{E}_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial z^i}, \quad (12)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j} + \tilde{w}^j \frac{\partial}{\partial w^j}$ and $(x^i, y^i, z^i, w^i, \tilde{w}^i)$ are the canonical local coordinates on $T^4 M$ induced by the local coordinates (x^i) on M . In the case when a second order Lagrangian L_ω is affine in accelerations and it is associated with a Pfaff form ω , its local formula is given as in formula (4) with $L^{(2)} = L_\omega$. We say that the Euler-Lagrange top Pfaff form $\mathcal{E} = \mathcal{E}_\omega$ of L_ω is the *Euler-Lagrange top Pfaff form* of ω . Specifically, if ω is a Pfaff form given by formula (1), then $L_\omega(t, x^i, y^i, z^i) = \omega_0 + y^i \omega_i + z^i \bar{\omega}_i$, thus

$$\mathcal{E}_i = \frac{\partial \omega_0}{\partial x^i} + \frac{\partial \omega_j}{\partial x^i} y^j + \frac{\partial \bar{\omega}_j}{\partial x^i} z^j - \frac{d}{dt} \left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} y^j + \omega_i + \frac{\partial \bar{\omega}_j}{\partial y^i} z^j \right) + \frac{d^2}{dt^2} \bar{\omega}_i, \quad (13)$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + w^j \frac{\partial}{\partial z^j}$, since $\frac{\partial L}{\partial z^i} = \bar{\omega}_i(t, x^i, y^i, z^i)$ and the forth order coordinates (\tilde{w}^i) are not involved. Thus the top Pfaff form \mathcal{E} is at most third order in this case.

We prove below that the Euler-Lagrange top Pfaff form can be obtained using two second order Pfaff forms.

Proposition 4.1 *Let ω be a (first order) Pfaff form such that the Euler-Lagrange top Pfaff form \mathcal{E}_ω is of third order. Then the following assertions holds true.*

1. *If Ω is a first or a second order Pfaff form such that $\bar{\Omega} = \bar{\omega}$, then there is a second or a third order Pfaff form Φ , uniquely determined by the conditions that the Lagrange top derivative of Ω is $\bar{\Phi}$ and the Lagrange top derivative of Φ is the Euler-Lagrange top Pfaff form \mathcal{E}_ω .*
2. *There are two second order Pfaff forms Ω and Φ such that $\bar{\Omega} = \bar{\omega}$, the Lagrange top derivative of Ω is $\bar{\Phi}$ and the Lagrange top derivative of Φ is \mathcal{E}_ω .*

Proof. 1. The conditions on Φ read $\bar{\Phi}_i = \Omega_i - \frac{d}{dt}\bar{\Omega}_i = \Omega_i - \frac{d}{dt}\bar{\omega}_i$ and $\Phi_i = \frac{d}{dt}\bar{\Phi}_i + \mathcal{E}_\omega$, thus Ω is uniquely determined by these conditions. If Ω is of first order then $\bar{\Phi}$ is of second order, thus Φ is of second or of third order. If Ω is of second order then $\bar{\Phi}$ is of second order or of third order, thus Φ is of second or of third order.

2. Let us denote $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i$ and let $L = \omega_0 + \omega_i y^i + \bar{\omega}_i z^i$ be the associated two order Lagrangian linear in velocities. We consider $\Omega = \Omega_i dx^i + \bar{\Omega}_i dx^i = \left(\frac{\partial L}{\partial y^i} - \omega_i\right) dx^i + \frac{\partial L}{\partial z^i} dx^i = \left(\frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} y^j + \frac{\partial \bar{\omega}_j}{\partial y^i} z^j\right) dx^i + \bar{\omega}_i dy^i$. We have $\frac{\partial}{\partial y^i} = \frac{\partial y^{i'}}{\partial z^{i'}} \frac{\partial}{\partial z^{i'}} + 2 \frac{\partial x^{i'}}{\partial x^i} \frac{\partial}{\partial y^{i'}}$ and $\frac{\partial}{\partial z^{i'}} = \frac{\partial x^{i'}}{\partial z^{i'}} \frac{\partial}{\partial z^{i'}}$. It follows that

$$\begin{aligned}\bar{\Omega}_i &= \frac{\partial x^{i'}}{\partial x^i} \bar{\Omega}_{i'}, \\ \Omega_i &= \frac{\partial y^{i'}}{\partial x^i} \bar{\Omega}_{i'} + \frac{\partial x^{i'}}{\partial x^i} \Omega_{i'},\end{aligned}\tag{14}$$

thus the local formulas $\Omega = \Omega_i dx^i + \bar{\Omega}_i dy^i$ define a second order Pfaff form $\Omega : \mathbb{R} \times T^2 M \rightarrow T^* T M$.

According to 1., $\bar{\Phi}_i = \Omega_i - \frac{d}{dt}\bar{\omega}_i = \frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i}$ and $\Phi_i = \frac{d}{dt}\bar{\Phi}_i + \mathcal{E}_\omega = \frac{\partial L}{\partial x^i} - \frac{d}{dt}\omega_i$, thus Φ is of second order. \square

We call the second order Pfaff form Ω constructed in 2. of Proposition 4.1 as an *Ostrogradski Pfaff form* of ω .

The above construction is related to a general approach, related to the classical Ostrogradski theory.

Let $L : \mathbb{R} \times T^2 M \rightarrow \mathbb{R}$ be a second order Lagrangian. There is a top Pfaff form $\bar{\omega} = \frac{\partial L}{\partial z^i} dx^i$ associated with this Lagrangian, that is of order at most 2. Let us suppose that ω is a first or second order Pfaff form ω such that its top Pfaff form is $\bar{\omega}$, i.e. $\omega = \omega_i dx^i + \frac{\partial L}{\partial z^i} dy^i$. One can consider for example $\omega = \frac{1}{2} \frac{\partial L}{\partial y^i} dx^i + \frac{\partial L}{\partial z^i} dy^i$. Then the formula $\Omega = \Omega_i dx^i + \bar{\Omega}_i dx^i = \left(\frac{\partial L}{\partial y^i} - \omega_i\right) dx^i + \frac{\partial L}{\partial z^i} dx^i$ defines a Pfaff form of order at most 2.

Then $\eta : \mathbb{R} \times T^2M \rightarrow T^*TM$, $\eta(t, x^i, y^i, z^i) = \left(\frac{\partial L}{\partial y^i} - \omega_i \right) dx^i + \frac{\partial L}{\partial z^i} dy^i$ is a second order Pfaff form. The Lagrange top derivative of η is $\mathcal{E}_\eta^{(3)} : \mathbb{R} \times T^2M \rightarrow T^*M$, $\mathcal{E}_\eta'' = \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right) dx^i$. Usually $\left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right)$ is denoted by p_i .

Since $\mathcal{E} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial y^i} + \frac{d^2}{dt^2} \frac{\partial L}{\partial z^i} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \omega_i - \frac{d}{dt} \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right)$, it follows that $\mu = \left(\frac{\partial L}{\partial x^i} - \frac{d}{dt} \omega_i \right) dx^i + \left(\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L}{\partial z^i} \right) dy^i$ is at most third order Pfaff form and its Lagrange top derivative is $\mathcal{E}_\mu^{(4)} = \mathcal{E}_\omega$, the Euler-Lagrange top Pfaff form of ω . This algorithm can produce Pfaff forms for a second order Lagrangian, taking a suitable Pfaff form ω .

A k -order semi-spray $S : \mathbb{R} \times T^kM \rightarrow T^{k+1}M$ is a section $(t, x^{(k)}) \xrightarrow{\bar{S}} (t, S(t, x^{(k)}))$ of the affine bundle $\mathbb{R} \times T^{k+1}M \rightarrow \mathbb{R} \times T^kM$, obtained as a product of the affine bundle $T^{k+1}M \rightarrow T^kM$ and the identity $\mathbb{R} \rightarrow \mathbb{R}$.

Let $\mathcal{E}_\omega : \mathbb{R} \times T^3M \rightarrow T^*M$ be the Euler-Lagrange top Pfaff form of a Pfaff form ω . We say that a second order semi-spray $\bar{S} : \mathbb{R} \times T^2M \rightarrow \mathbb{R} \times T^3M$ is adapted to the Pfaff form ω if $\mathcal{E}_\omega \circ \bar{S} = 0$. The local form of \bar{S} and \mathcal{E}_ω are $(t, x^i, y^i, z^i) \xrightarrow{\bar{S}} (t, x^i, y^i, z^i, S^i(t, x^i, y^i, z^i))$ and $\mathcal{E}_\omega = \mathcal{E}_i dx^i$ with \mathcal{E}_i given by (13) respectively. Denoting $h_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i}$, then there are local functions $f_i(t, x^i, y^i, z^i)$ such that

$$\mathcal{E}_i(t, x^i, y^i, z^i, w^i) = h_{ij} w^j + f_i(t, x^i, y^i, z^i).$$

More precisely:

$$\begin{aligned} \mathcal{E}_i = & \frac{\partial \omega_0}{\partial x^i} - \frac{\partial^2 \omega_0}{\partial t \partial y^i} - \frac{\partial^2 \omega_0}{\partial x^j \partial y^i} y^j + \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial^2 \omega_j}{\partial t \partial y^i} - \frac{\partial \omega_j}{\partial x^k \partial y^i} - \right. \\ & \left. \frac{\partial \omega_j}{\partial y^j \partial y^i} \right) y^j - \frac{\partial \omega_i}{\partial t} - \frac{\partial \omega_i}{\partial x^j} y^j + \frac{\partial^2 \bar{\omega}_i}{\partial t^2} + \frac{\partial^2 \bar{\omega}_i}{\partial x^j \partial t} y^j + \left(\frac{\partial^2 \bar{\omega}_i}{\partial t \partial x^j} + \frac{\partial^2 \bar{\omega}_i}{\partial x^k \partial x^j} y^k \right) y^j - \\ & - \frac{\partial^2 \omega_0}{\partial y^j \partial y^i} z^j - \frac{\partial \omega_j}{\partial y^i} z^j - \frac{\partial \omega_i}{\partial y^j} z^j + \frac{\partial \bar{\omega}_j}{\partial x^i} z^j + \left(2 \frac{\partial^2 \bar{\omega}_i}{\partial x^k \partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^k \partial y^i} \right) y^k z^j + \\ & + \left(2 \frac{\partial^2 \bar{\omega}_i}{\partial y^j \partial t} - \frac{\partial^2 \bar{\omega}_j}{\partial t \partial y^i} \right) z^j + \frac{\partial \bar{\omega}_i}{\partial x^j} z^j + \left(\frac{\partial^2 \bar{\omega}_i}{\partial y^k \partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^k \partial y^i} \right) z^k z^j + \left(\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right) w^j. \end{aligned}$$

The condition that \bar{S} is adapted to the Pfaff form ω reads

$$h_{ij} S^j + f_i(t, x^i, y^i, z^i) = 0. \quad (15)$$

Notice that the third order semi-spray S gives a system of third order equations, having the form

$$\frac{d^3 x^i}{dt^3} + S^i(t, x^i, \frac{dx^i}{dt}, \frac{d^2 x^i}{dt^2}) = 0;$$

its solutions are the integrable curves of the vector field S .

We say that the Pfaff form ω is *regular* if the local matrices $(h_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i})$ are non-singular.

Proposition 4.2 *If the Pfaff form ω is regular, then the solutions of the generalized Euler-Lagrange equation $\mathcal{E} = 0$, where \mathcal{E} is given by (13) are the same solutions of a second order equation given by a global second order semi-spray $S : \mathbb{R} \times T^2M \rightarrow T^3M$.*

Proof. If ω is regular, then the matrix (h_{ij}) is invertible and let $(h^{ij}) = (h_{ij})^{-1}$. The equation (15) gives uniquely $S^j = h^{ji} f_i(t, x^i, y^i, z^i)$ and a map $S : \mathbb{R} \times T^2 M \rightarrow T^3 M$, $S(t, x^i, y^i, z^i) = (t, x^i, y^i, z^i, S^i)$, that is well defined and gives the second order semi-spray S . \square

5 Non-degenerated and regular Pfaff forms

We recall that a (first order) Pfaff form ω having the local form (1) is

regular if the local matrix $\left(h_{ij} = \frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j}\right)$ is non-singular and

non-degenerated if the local matrix $\left(\frac{\partial \bar{\omega}_i}{\partial y^j}\right)$ is non-singular.

The two above regularity conditions can be related as follows.

If $m = \dim M$ is odd, then there are not regular Pfaff form on M , since a skew symmetric matrix is singular in this case; but there are non-degenerated Pfaff forms. For example, let g be a (pseudo-) Riemannian metric on M , $F(x, y) = \frac{1}{2}g_x(y, y)$ its energy map and $\bar{\omega}_i = \frac{\partial F}{\partial y^i}$. Then any Pfaff form that has $\bar{\omega} = \bar{\omega}_i dx^i$ a top Pfaff form is non-degenerated. Even in even dimension, this top Pfaff form is non-degenerated, but never regular, since $\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} = 0$.

In the even dimensions, there are regular Pfaff forms that are degenerated. For example, in \mathbb{R}^2 , the top Pfaff form $\bar{\omega}(x, y, X, Y) = (X + Y)dx - (X - Y)dy$ is regular, but degenerated.

We say that a Pfaff ω form is *biregular* if it is hyper-non-degenerated as well as regular.

Let us extend the definition of a non-degenerated Pfaff form to higher order Pfaff forms and use it in the study of the Euler-Lagrange equation.

For $k \geq 1$, let us denote $T^{k*}M = T^*M \times_M T^{k-1}M$, the fibered product over the base M . A k -order top Pfaff form $\bar{\omega} : \mathbb{R} \times T^k M \rightarrow \mathbb{R} \times T^*M$ gives rise to a bundle map $\mathcal{L}_{\bar{\omega}} : \mathbb{R} \times T^k M \rightarrow \mathbb{R} \times T^{k*}M$, $\mathcal{L}_{\bar{\omega}}(t, x^{(k)}) = (t, \pi_k(x^{(k)}), \bar{\omega}(t, x^{(k)}))$, that we call the *Legendre map* of $\bar{\omega}$. The *Legendre map* \mathcal{L}_{ω} of a k -order Pfaff form ω is, by definition, the Legendre map of its associated top form. The condition that $\mathcal{L}_{\bar{\omega}}$ be a local diffeomorphism is just that $\bar{\omega}$ be a *non-degenerated* top Pfaff form. We say that $\bar{\omega}$ is *hyper-non-degenerated* if $\mathcal{L}_{\bar{\omega}}$ is a global diffeomorphism. The same definitions (*Legendre map*, *non-degenerated*, *hyper-non-degenerated* and *biregular*) on a Pfaff form ω are the same as referring to its top form $\bar{\omega}$, as above.

We recall that a k -order semi-spray on M is a section $S : \mathbb{R} \times T^k M \rightarrow \mathbb{R} \times T^{k+1}M$ of the affine bundle $\mathbb{R} \times T^{k+1}M \xrightarrow{\pi_k} \mathbb{R} \times T^k M$. It can be regarded as well as a (time dependent) vector field Γ_0 on the manifold $T^k M$, since $T^{k+1}M \subset TT^k M$.

Let $\pi_E : E \rightarrow M$ be a fibered manifold. For $k \geq 1$, we say that a *controlled semi-spray of degree k* on M over E is a map $\bar{S} : \mathbb{R} \times E \times_M T^k M \rightarrow T^{k+1}M$ such $\pi_k \circ \bar{S}(t, e, x^{(k)}) = x^{(k)}$. We use here in particular a controlled semi-spray of order k over $E = T^*M$. We denote $T^{k*}M = T^*M \times_M T^{k-1}M$, considered as a fibered

manifold over M and we say that a controlled semi-spray $\bar{S} : \mathbb{R} \times T^{k*}M \rightarrow T^k M$ is $(k-1)$ -order cotangent semi-spray, or a $(k-1)$ -co-semi-spray, in short.

If a top Pfaff form of order k is hyper-non-degenerated, then the inverse of the Legendre map $\mathcal{L}_{\bar{\omega}}, \mathcal{L}_{\bar{\omega}}^{-1} : \mathbb{R} \times T^{k*}M \rightarrow \mathbb{R} \times T^k M$ defines a $(k-1)$ -co-spray $\bar{S} : \mathbb{R} \times T^{k*}M \rightarrow T^k M$.

For example, if a (first order) top Pfaff form, or Pfaff form, is hyper-non-degenerated, then the inverse of its Legendre map defines a 1-semi-spray $\bar{S} : \mathbb{R} \times T^*M \rightarrow TM$.

5.1 The dynamics of regular and biregular Pfaff forms

We prove in this subsection that the dynamics on M of a regular Pfaff form comes from the projection of the integral curves of a vector field X on $T^{2*}M = T^*M \times_M TM$, while for a biregular Pfaff form, its dynamics comes from the projection of the integral curves of a vector field Y on $T_2^0 M = TM \times_M TM$.

A regular (first order) Pfaff form ω gives rise to a 2-co-semi-spray $\bar{S} : \mathbb{R} \times T^{2*}M \rightarrow T^2 M$ as follows. Let us consider the Ostrogradski Pfaff form $\Omega = (\frac{\partial L_{\omega}}{\partial y^i} - \omega_i)dx^i + \frac{\partial L_{\omega}}{\partial z^i}dx^i$, $L_{\omega} = \omega_0 + \omega_i y^i + \bar{\omega}_i z^i$, Φ be the second order Pfaff form given by Proposition 4.1 and $\Phi = \Phi_i dx^i + \bar{\Phi}_i dy^i$. We have

$$\begin{aligned} \bar{\Phi}_i &= \frac{\partial L_{\omega}}{\partial y^i} - \omega_i - \frac{d}{dt} \frac{\partial L_{\omega}}{\partial z^i} = \frac{\partial \omega_0}{\partial y^i} + \frac{\partial \omega_j}{\partial y^i} y^j + \frac{\partial \bar{\omega}_i}{\partial y^i} z^j - \left(\frac{\partial \bar{\omega}_i}{\partial x^j} y^j + \frac{\partial \bar{\omega}_i}{\partial y^j} z^j + \frac{\partial \omega_i}{\partial t} \right) = \\ &= \frac{\partial \omega_0}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial t} + \left(\frac{\partial \omega_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial x^j} \right) y^j + \left(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j} \right) z^j \text{ and} \\ \Phi_i &= \frac{\partial L_{\omega}}{\partial x^i} - \frac{d}{dt} \omega_i = \frac{\partial \omega_0}{\partial x^i} - \frac{\partial \omega_i}{\partial t} + \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) y^j + \left(\frac{\partial \bar{\omega}_j}{\partial x^i} - \frac{\partial \omega_i}{\partial y^j} \right) z^j. \end{aligned}$$

Let us consider the map $\mathcal{L}' : \mathbb{R} \times T^2 M \rightarrow \mathbb{R} \times T^{2*}M$, $\mathcal{L}'(t, x^{(2)}) = (t, \pi_1(x^{(2)}), \mathcal{E}_{\Omega}^{(2)}(x^{(2)}))$, where $\mathcal{E}_{\Omega}^{(2)} : T^2 M \rightarrow T^{2*}M$ is the Lagrange top derivative of Ω .

It is easy to see that ω is a regular Pfaff form, i.e. the matrix $\left(\frac{\partial \bar{\omega}_j}{\partial x^i} - \frac{\partial \omega_i}{\partial y^j} \right)$ is non-singular, iff \mathcal{L}'_{ω} is a global diffeomorphism. Then the inverse of \mathcal{L}'_{ω} gives a 2-co-semi-spray $S : \mathbb{R} \times T^{2*}M \rightarrow T^2 M$. The local form of the functions S^i comes from the equations $\bar{\Phi}_i = p_i$, thus

$$S^j = h^{ij} \left(p_i - \frac{\partial \omega_0}{\partial y^i} + \frac{\partial \bar{\omega}_i}{\partial t} - \left(\frac{\partial \omega_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial x^j} \right) y^j \right), \quad (16)$$

where $(h^{ij}) = (h_{ij})^{-1}$, $h_{ij} = \frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j}$.

Now we can go further, to find the integral curves of the action of ω .

Considering local coordinates, then $\mathcal{E}'_{\omega} = \Omega = \Omega_i dx^i + \bar{\Omega}_i dy^i$ and let $X = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + S^i(t, x^j, y^j, p_j) \frac{\partial}{\partial y^i} + \bar{\Phi}_i \frac{\partial}{\partial p_i}$ be the (t, x^i, y^i) local form of local vector fields on $\mathbb{R} \times T^{2*}M$, where $\bar{\Phi}_i(t, x^i, y^i, p_i) = \Phi_i(t, x^i, y^i, S^i(t, x^i, y^i, p_i)) = \frac{\partial \omega_0}{\partial x^i} - \frac{\partial \omega_i}{\partial t} + \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) y^j + \left(\frac{\partial \bar{\omega}_j}{\partial x^i} - \frac{\partial \omega_i}{\partial y^j} \right) S^j$. Let us denote by $\pi_{*2} : T^{2*}M \rightarrow M$ the canonical projection.

Proposition 5.1 *Let ω be a regular (first order) Pfaff form. Then:*

1. the local vector fields glue together to a vector field X on $T^{2*}M$ and

2. the integral curves of X projects by π_{*2} to all the critical curves of the action of ω .

Proof. We use local coordinates (x^i) , (x^i, y^i) , (x^i, p_i) , (x^i, y^i, p_i) and $(x^i, y^i, p_i, X^i, Y^i, P_i)$ on M , TM , T^*M , $T^{2*}M = TM \times_M T^*M$ and $T(TM \times_M T^*M)$ respectively (see the Appendix). Then the local components of X must change on the intersection of two local bundle charts by the rules $S^{i'} = \frac{\partial y^{i'}}{\partial x^i} y^i + \frac{\partial x^{i'}}{\partial x^i} S^i$ and $\tilde{\Phi}_i = \frac{\partial y^{i'}}{\partial x^i} p_{i'} + \frac{\partial x^{i'}}{\partial x^i} \tilde{\Phi}_{i'}$ respectively. The first rule follows from the fact that (S^i) are the components of a 1-co-semi-spray. The second rule follows using similar relations 14 for Φ_i and $\tilde{\Phi}_i$: in the second relation we have that $\tilde{\Phi}_{i'} = p_{i'}$. Thus 1. follows.

Along an integral curve of X we have $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = S^i$ and $\frac{dp_i}{dt} = \tilde{\Phi}_i$. Since $p_i = \tilde{\Phi}_i(t, x^j, y^j, S^j)$ and $\tilde{\Phi}_i = \frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \bar{\omega}_i$, thus $\frac{dp_i}{dt} = \frac{d}{dt} (\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \bar{\omega}_i)$, it follows that $\tilde{\Phi}_i = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \omega_i = \frac{d}{dt} (\frac{\partial L}{\partial y^i} - \omega_i - \frac{d}{dt} \bar{\omega}_i)$, i.e. the Euler-Lagrange equation holds along any curve $t \rightarrow x(t)$. This proves 2. \square

Let us consider that ω is biregular, i.e. hyper-non-degenerated and regular. Let us denote $T_2^0 M = T^*M \times_M T^*M$ and $\pi_{02*} : T_2^0 M \rightarrow M$ the canonical projection and consider coordinates $(x^i, p_{(0)i}, p_{(1)i})$ on $T_2^0 M$, induced by the local coordinates (x^i) on M . We define a map $\mathcal{L}''_\omega : \mathbb{R} \times T^2 M \rightarrow \mathbb{R} \times T_2^0 M$, $\mathcal{L}''_\omega(t, x^{(2)}) = (t, \bar{\Omega}, \bar{\Phi})$, where $\bar{\Omega} = \bar{\omega}$ and $\bar{\Phi}$ are the corresponding top Pfaff forms.

Then ω is non-degenerated and regular iff the map \mathcal{L}''_ω is a local diffeomorphism. The Pfaff form ω is hyper-non-degenerated iff \mathcal{L}''_ω is a diffeomorphism.

Considering local coordinates as previously, the Legendre map \mathcal{L}''_ω has the form $\mathcal{L}''_\omega(t, x^i, y^i, z^i) = (t, \bar{\omega}_i(t, x^i, y^i), \bar{\Omega}_i(t, x^i, y^i, z^i))$. As above it follows that \mathcal{L}''_ω is a local diffeomorphism that is a global one iff the Legendre map is a global diffeomorphism.

Let $(t, p_{(0)i}, p_{(1)i})$ be some local coordinates on $\mathbb{R} \times T_2^0 M$. If the Pfaff form ω is biregular, then the equations $\bar{\omega}_i(t, x^i, y^i) = p_{(0)i}$ gives $y^i = T^i(t, x^j, p_{(0)j})$ and the equations $\bar{\Omega}_i(t, x^i, T^i, z^i) = p_{(1)i}$ gives $z^i = S^i(t, x^j, p_{(0)j}, p_{(1)j})$. If ω is hyperregular and hyper-non-degenerated, then the local functions (T^i) and (S^i) come from some co-semi-sprays and give some global diffeomorphisms $T : \mathbb{R} \times T^*M \rightarrow \mathbb{R} \times TM$ and $S : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R} \times T^2 M$ respectively.

Let us consider the local vector field on $T_2^0 M$, given by $Y = T^i \frac{\partial}{\partial x^i} + (\hat{\Omega}_i - p_{(1)i}) \frac{\partial}{\partial p_{(0)i}} + \hat{\Phi}_i \frac{\partial}{\partial p_{(1)i}}$, where $\hat{\Omega}_i(t, x^j, p_{(0)j}) = \Omega_i(t, x^j, T^j)$, $\hat{\Phi}_i(t, x^j, p_{(0)j}, p_{(1)j}) = \Phi_i(t, x^j, T^j, S^j(t, x^j, T^j, p_{(1)j}))$.

Proposition 5.2 *Let ω be a biregular (first order) Pfaff form. Then:*

1. the local vector fields glue together to a vector field Y on $T_2^0 M$ and
2. the integral curves of Y projects by π_{*2} to all the critical curves of the action of ω .

Proof. We use local coordinates (x^i) , $(x^i, p_{(0)i}, p_{(1)i})$ and $(x^i, p_{(0)i}, p_{(1)i}, x^i, P_{(0)i}, P_{(1)i})$ on M , $T_2^0 M = T^*M \times_M T^*M$ and $T(TM^* \times_M T^*M)$ respectively

(see the Appendix). Then the local components of Y must change on the intersection of two local bundle charts by the rules $T^{i'} = \frac{\partial x^{i'}}{\partial x^i} T^i$, $(\hat{\Omega}_i - p_{(1)i}) = \frac{\partial x^{i'}}{\partial x^i} (\hat{\Omega}_{i'} - p_{(1)i'}) + \frac{\partial y^{i'}}{\partial x^i} p_{(0)i}$ and $\hat{\Phi}_i = \frac{\partial x^{i'}}{\partial x^i} \hat{\Phi}_{i'} + \frac{\partial y^{i'}}{\partial x^i} p_{(1)i}$. The first relation follows from the fact that (T^i) are the components of a global map in the fibers of TM . The second relation follows from $p_{(1)i} = \frac{\partial x^{i'}}{\partial x^i} p_{(1)i'}$ and $\Omega_i = \frac{\partial x^{i'}}{\partial x^i} \Omega_{i'} + \frac{\partial y^{i'}}{\partial x^i} p_{(0)i}$ (see the Appendix and the definition of $\hat{\Omega}$). The third relation follows using the definition of $\hat{\Theta}$, the second relation 14 and the fact that $\bar{\Phi}_{i'} = p_{(1)i'}$. In order to prove 2., along an integral curve of Y we have $y^i = \frac{dx^i}{dt} = T^i$, $\frac{dp_{(0)i}}{dt} = \hat{\Omega}_i - p_{(1)i}$ and $\frac{dp_{(1)i}}{dt} = \hat{\Theta}_i$. According to the definitions, it is easy to prove 2. \square

A full interpretation of the two vector fields X and Y is given in the next subsection, where we prove that the two vectors are the Hamiltonian vector fields of two suitable Hamiltonians.

5.2 Hamiltonian descriptions of biregular Pfaff forms

Important tools in describing the dynamic equations of a Hamiltonian system are offered by quantizations. Following similar ideas used in [9, Section 2.], one can use Ostrogradski-Dirac and Fadeev-Jakiw methods, but also a modified Ostrogradski-Dirac method, according to the possibility to construct constraints slight different from the canonical ones used in Ostrogradski theory. The Ostrogradski-Dirac method was also used in [3] in the quantization of the system derived from a Lagrangian linear in velocities, involved in the study of a Reegge-Teitelboim model. We give below a global form of these results. More specifically, we prove in this subsection that:

- if ω is regular and its essential part is time independent, then there are symplectic forms Ξ'_t on $T^{2*}M$, $t \in \mathbb{R}$, and a hamiltonian $H : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field X_H gives by projection the dynamics of ω on M ;
- if ω is biregular and its essential part is time independent, then there are symplectic forms Ξ''_t on $T_2^0 M$, $t \in \mathbb{R}$, and a hamiltonian $H' : \mathbb{R} \times T_2^0 M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field $X_{H'}$ gives by projection the dynamics of ω on M .

Let us consider the map $\Phi : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R} \times T^*TM$, $\Phi(t, x^i, y^i, p_i) = (t, x^i, y^i, p_i + \omega_i, \bar{\omega}_i)$. Let us denote by $\Phi_t : T^{2*}M \rightarrow T^*TM$ the map $\Phi_t(p^{(2)}) = \Phi(t, p^{(2)})$, where $t \in \mathbb{R}$ is given, and by Ξ the canonical symplectic 2-form on T^*TM . Then we can consider the induced 2-form $\Phi_t^* \Xi$ on $T^{2*}M$, that has the local form $\Phi_t^* \Xi = dx^i \wedge (dp_i + d\omega_i) + dy^i \wedge d\bar{\omega}_i$, where the differential d is considered on $T^{2*}M$.

Proposition 5.3 *Let ω be a (first order) Pfaff form. For every $t \in \mathbb{R}$ the form $\Xi'_t = \Phi_t^* \Xi$ is closed on $T^{2*}M$ and it is non-degenerated iff ω is a regular Pfaff form.*

Proof. The form $\Phi_t^* \Xi$ is closed since the form Ξ is closed. Using local coordinates as above, we have:

$\Phi_t^* \Xi = dx^i \wedge (dp_i + \frac{\partial \omega_i}{\partial x^j} dx^j + \frac{\partial \omega_i}{\partial y^j} dy^j) + dy^i \wedge (\frac{\partial \bar{\omega}_i}{\partial x^j} dx^j + \frac{\partial \bar{\omega}_i}{\partial y^j} dy^j) = dx^i \wedge dp_i + \frac{\partial \omega_i}{\partial x^j} dx^i \wedge dx^j + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \bar{\omega}_i}{\partial x^i} \right) dx^i \wedge dy^j + \frac{\partial \bar{\omega}_i}{\partial y^j} dy^i \wedge dy^j$. Thus, using the local base $\{dx^i \wedge dx^j, dx^i \wedge dy^j, dx^i \wedge dp_j, dy^i \wedge dy^j, dy^i \wedge dp_j, dp_i \wedge dp_j\}_{i < j}$, then $\Phi_t^* \Xi$ has the matrix

$$\begin{pmatrix} A & B & I \\ -B & C & 0 \\ -I & 0 & 0 \end{pmatrix}$$

where $A = \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right)$, $B = \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^i} \right)$, $I = (\delta_j^i)$, $C = \left(\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right)$. The above matrix is non-degenerate iff the matrix C is non-degenerate, i.e. iff ω is a regular Pfaff form. \square

We prove now that the closed form Ξ' can be used to quantify the Hamiltonian system derived from a Lagrangian linear in velocities that comes from a non-degenerate Pfaff form.

Theorem 5.1 *Let ω be a regular (first order) Pfaff form on M such that its essential part is time independent. Then there is are symplectic forms Ξ'_t on $T^{2*}M$, $t \in \mathbb{R}$, and a hamiltonian $H : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field X_H is X from Proposition 5.1.*

Proof. According to Proposition 5.1, it suffices to prove that the vector field $X = \frac{\partial}{\partial t} + y^i \frac{\partial}{\partial x^i} + S^i(t, x^i, y^i, p_i) \frac{\partial}{\partial y^i} + \tilde{\Phi}_i \frac{\partial}{\partial p_i}$ is the Hamiltonian vector field of a suitable Hamiltonian, namely the Hamiltonian $H = -p_i y^i + \omega_0$. It reads that $i_X \Xi = dH$. Indeed, using local coordinates, we have:

$$\begin{aligned} & \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) y^j + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^i} \right) S^j + \delta_i^j \tilde{\Phi}_j = \\ & \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) y^j + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^i} \right) S^j + \\ & \frac{\partial \omega_0}{\partial x^i} - \frac{\partial \omega_i}{\partial t} + \left(\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) y^j + \left(\frac{\partial \bar{\omega}_i}{\partial x^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right) S^j = \\ & \frac{\partial \omega_0}{\partial x^i} - \frac{\partial \omega_i}{\partial t} = \frac{\partial \omega_0}{\partial x^i} = \frac{\partial H}{\partial x^i}, \\ & - \left(\frac{\partial \omega_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial x^j} \right) y^j + \left(\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right) S^j = \\ & - \left(\frac{\partial \omega_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial x^j} \right) y^j - p_i + \frac{\partial \omega_0}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial t} + \left(\frac{\partial \omega_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial x^i} \right) y^j = \\ & - p_i + \frac{\partial \omega_0}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial t} = -p_i + \frac{\partial \omega_0}{\partial y^i} = \frac{\partial H}{\partial y^i} \text{ and} \\ & -\delta_i^j y^j = -y^i = \frac{\partial H}{\partial p_i}. \quad \square \end{aligned}$$

In the case when the essential part $\tilde{\omega} = \omega_i dx^i + \bar{\omega}_i dy^i$ of ω is not necessarily time independent, the general formula reads $i_X \Xi = dH - \frac{\partial}{\partial t} \tilde{\omega}$, where $\frac{\partial}{\partial t} \tilde{\omega} = \frac{\partial \omega_i}{\partial t} dx^i + \frac{\partial \bar{\omega}_i}{\partial t} dy^i$ is a 1-form on $T^{2*}M$ induced by the canonical projection $T^{2*}M \rightarrow TM$, by a 1-form given by the same formula.

Let $\bar{\omega} = \bar{\omega}_i dx^i$ be a hyper-non-degenerated (first order) top Pfaff form, i.e. the Legendre map $\mathcal{L}_{\bar{\omega}} : \mathbb{R} \times TM \rightarrow \mathbb{R} \times T^*M$ is a global diffeomorphism. Then $\mathcal{L}_{\bar{\omega}}^{-1} : \mathbb{R} \times T^*M \rightarrow \mathbb{R} \times TM$ has the form $(t, x^i, p_i) \xrightarrow{\mathcal{L}_{\bar{\omega}}^{-1}} (t, x^i, T^i(t, x^i, p_i))$. Considering the non-degenerated matrices $(h_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j})$ and its inverse $(\bar{h}^{ij} = \frac{\partial T^i}{\partial p_j})$,

we say that $\bar{\omega}$ is co-regular if the matrix $\left(\tilde{h}^{ij} = \frac{\partial T^i}{\partial p_j} - \frac{\partial T^j}{\partial p_i}\right)$ is non-singular in every point of $\mathbb{R} \times T^*M$. We recall that $\bar{\omega}$ is regular if the matrix

$$\left(\tilde{h}_{ij} = \frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i}\right)$$

is non-singular in every point of $\mathbb{R} \times TM$.

We say that a Pfaff form ω is co-regular if its top Pfaff form $\bar{\omega}$ is co-regular.

Proposition 5.4 *If a Pfaff form ω is non-degenerated then ω is co-regular iff ω is regular.*

Proof. Denoting $H = (h_{ij})$, $\bar{H} = (\bar{h}^{ij}) = H^{-1}$, then $\left(\tilde{h}^{ij}\right) = \bar{H} - \bar{H}^t = \bar{H}(H^t - H)\bar{H}^t$, thus $\left(\tilde{h}^{ij}\right) = \bar{H} - \bar{H}^t$ is invertible iff $(h_{ij} - h_{ji}) = H - H^t$ is invertible; this prove the assertion. \square

Let us suppose that the Pfaff form ω is biregular, i.e. hyper-non-degenerated and regular. Thus there are some global co-semi-sprays that give some global diffeomorphisms $T : \mathbb{R} \times T^*M \rightarrow \mathbb{R} \times TM$ and $S : \mathbb{R} \times T^{2*}M \rightarrow \mathbb{R} \times T^2M$ respectively. We consider the local functions (T^i) and (S^i) that come from these co-semi-sprays.

Let us consider the diffeomorphism $\Psi : \mathbb{R} \times T_2^0M \rightarrow \mathbb{R} \times T^*TM$,

$$\Psi(t, x^i, p_{(0)i}, p_{(1)i}) = (t, x^i, T^i(t, x^j, p_{(0)j}), p_{(1)i} + \tilde{\omega}_i(t, x^j, p_{(0)j}), p_{(0)i}),$$

where $\tilde{\omega}_i(t, x^j, p_{(0)j}) = \omega_i(t, x^j, T^j(t, x^j, p_{(0)j}))$. Let us denote by $\Psi_t : T_2^0M \rightarrow T^*TM$ the map $\Psi_t(x^i, p_{(0)i}, p_{(1)i}) = \Psi(t, x^i, p_{(0)i}, p_{(1)i})$, where $t \in \mathbb{R}$ is given, and by Ξ the canonical symplectic 2-form on T^*TM . Then we can consider the induced 2-form $\Psi_t^*\Xi$ on T_2^0M , that has the local form $\Phi_t^*\Xi = dx^i \wedge (p_{(1)i} + d\tilde{\omega}_i) + dT^i \wedge dp_{(0)i}$, where the differential d is considered on T_2^0M .

Let us denote by $F : \mathbb{R} \times T_2^0M \rightarrow \mathbb{R} \times T^{2*}M$ the diffeomorphism given by $F(t, x^i, p_{(0)i}, p_{(1)i}) = (t, x^i, T^i(t, x^j, p_{(0)j}), p_{(1)i})$, provided that ω is hyper-non-degenerated. It is easy to see that $\Psi_t = \Phi_t \circ F_t$, thus $\Psi_t^*\Xi = F_t^*\Phi_t^*\Xi = F_t^*\Xi'$. In a similar way as Proposition 5.3, the following statement holds true.

Proposition 5.5 *Let ω be a biregular Pfaff form. For every $t \in \mathbb{R}$ the two form $\Xi_t'' = \Psi_t^*\Xi = F_t^*\Xi'$ is symplectic form on T_2^0M .*

Using local coordinates as above, we have:

$$\Psi_t^*\Xi = dx^i \wedge (dp_{(1)i} + \frac{\partial \tilde{\omega}_i}{\partial x^j} dx^j + \frac{\partial \tilde{\omega}_i}{\partial p_{(0)j}} dp_{(0)j}) + (\frac{\partial T^i}{\partial x^j} dx^j + \frac{\partial T^i}{\partial p_{(0)j}} dp_{(0)j}) \wedge dp_{(0)i} =$$

$$dx^i \wedge dp_{(1)i} + \frac{\partial \tilde{\omega}_i}{\partial x^j} dx^i \wedge dx^j + \left(\frac{\partial \tilde{\omega}_i}{\partial p_{(0)j}} - \frac{\partial T^j}{\partial x^i}\right) dx^i \wedge dp_{(0)j} + \frac{\partial T^j}{\partial p_{(0)i}} dp_{(0)i} \wedge dp_{(0)j}.$$

Thus, using the local base $\{dx^i \wedge dx^j, dx^i \wedge dp_{(0)j}, dx^i \wedge dp_{(1)j}, dp_{(0)i} \wedge dp_{(0)j}, dp_{(0)i} \wedge dp_{(1)j}, dp_{(1)i} \wedge dp_{(1)j}\}_{i < j}$, then $\Phi_t^*\Xi$ has the matrix

$$\begin{pmatrix} A' & B' & I \\ -B' & C' & 0 \\ -I & 0 & 0 \end{pmatrix},$$

where $A' = \left(\frac{\partial \bar{\omega}_i}{\partial x^j} - \frac{\partial \bar{\omega}_j}{\partial x^i} \right)$, $B' = \left(\frac{\partial \bar{\omega}_i}{\partial p_{(0)j}} - \frac{\partial T^j}{\partial x^i} \right)$, $C' = \left(\frac{\partial T^j}{\partial p_{(0)i}} - \frac{\partial T^i}{\partial p_{(0)j}} \right)$ and $I = (\delta_{ij})$. The above matrix is non-degenerated iff the matrix C' is non-degenerate i.e. iff ω is a biregular Pfaff form.

We prove now that the closed form Ξ'' can be used also to quantify the Hamiltonian system derived from a Lagrangian linear in velocities that comes from a non-degenerate Pfaff form. Using Theorem 5.1, the following statement holds true.

Theorem 5.2 *Let ω be a biregular (first order) Pfaff form on M such that its essential part is time independent. Then for every $t \in \mathbb{R}$, there is a symplectic form Ξ_t'' on $T_2^0 M$ and a hamiltonian $H' : \mathbb{R} \times T_2^0 M \rightarrow \mathbb{R}$ such that the Hamiltonian vector field $X_{H'}$ is Y from Proposition 5.2.*

Proof. It suffices to prove that for every $t \in \mathbb{R}$, the vector fields $X = X_H \in \mathcal{X}(T^{2*}M)$ used in Theorem 5.1 and $Y \in \mathcal{X}(T_2^0 M)$ used in Proposition 5.2 are related by the diffeomorphisms $F_t : T_2^0 M \rightarrow T^{2*}M$, i.e. $(F_t)_* Y \circ (F_t^{-1}) = X$, or $(F_t^{-1})_* X \circ (F_t) = Y$.

Indeed, using local coordinates, $(F_t^{-1})_*$ has the local matrix

$$\begin{pmatrix} I & 0 & 0 \\ D & E & 0 \\ 0 & 0 & I \end{pmatrix},$$

where $I = (\delta_j^i)$, $D = \left(\frac{\partial \bar{\omega}_i}{\partial x^j} \right)$, $E = \left(\frac{\partial \bar{\omega}_i}{\partial y^j} \right)$. Then $(F_t^{-1})_* X$ and $Y' = (F_t^{-1})_* X \circ (F_t)$ have the local forms

$$\begin{aligned} (F_t^{-1})_* X &= y^i \frac{\partial}{\partial x^i} + (y^j \frac{\partial \bar{\omega}_i}{\partial x^j} + S^j \frac{\partial \bar{\omega}_i}{\partial y^j}) \frac{\partial}{\partial p_{(0)i}} + \bar{\Phi}_i \frac{\partial}{\partial p_{(1)i}} \text{ and} \\ Y' &= T^i \frac{\partial}{\partial x^i} + (T^i \frac{\partial \bar{\omega}_i}{\partial x^j}(t, x^i, T^i) + S^j(t, x^j, T^j, p_{(1)j}) \frac{\partial \bar{\omega}_i}{\partial y^j}(t, x^i, T^i)) \frac{\partial}{\partial p_{(0)i}} + \hat{\Phi}_i \frac{\partial}{\partial p_{(1)i}} = \\ &= T^i \frac{\partial}{\partial x^i} + (\hat{\Omega}_i - p_{(1)i}) \frac{\partial}{\partial p_{(0)i}} + \hat{\Phi}_i \frac{\partial}{\partial p_{(1)i}}, \text{ since} \\ &= y^i \frac{\partial \bar{\omega}_i}{\partial x^j} + S^j \frac{\partial \bar{\omega}_i}{\partial y^j} = \frac{\partial L_i}{\partial y^j} - \omega_i - p_i = \Omega_i - p_i. \end{aligned}$$

Thus $Y' = Y \in \mathcal{X}(T_2^0 M)$, used in Proposition 5.2.

Notice that the pull-back of the Hamiltonians $H = -p_i y^i + \omega_0$ by F_t is the Hamiltonian $H' : \mathbb{R} \times T_2^0 M \rightarrow \mathbb{R}$, $H'(t, x^i, p_{(0)i}, p_{(1)i}) = -p_{(1)i} T^i(x^j, p_{(0)j}) + \omega_0(t, x^i, T^i)$. \square

6 Some examples and special cases

We say that a Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ is *singular* if it is locally equivalent to a local Lagrangian.

Proposition 6.1 *A Pfaff form is singular iff its top component $\bar{\omega}$, viewed as a vertical form, is vertical closed.*

Proof. The Pfaff form $\omega = \omega_0 dt + \omega_i dx^i + \bar{\omega}_i dy^i = (\omega_0 + y^i \omega_i) dt + \omega_i \delta x^i + \bar{\omega}_i dy^i$ is locally equivalent to a local Lagrangian form iff locally its top component $\bar{\omega}_i$

has the form $\bar{\omega}_i = \frac{\partial \mu}{\partial y^i}$. Using Poincaré Lemma, this condition is equivalent to $\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} = 0$, i.e. $\bar{\omega} = \bar{\omega}_i dx^i$ is vertically closed. \square

We say that ω is:

globally singular if there are two Lagrangians $L_0, L_1 : \mathbb{R} \times TM \rightarrow \mathbb{R}$ and a top Lagrangian form $\mu = \mu_i dx^i$, such that $\omega - L_0 dt = \mu_i \delta x^i + dL_1$;

locally singular if there is a Lagrangian $L_0 : \mathbb{R} \times TM \rightarrow \mathbb{R}$, a closed form $\omega_0 \in \mathcal{X}^*(\mathbb{R} \times TM)$ and a top Lagrangian form $\mu = \mu_i dx^i$, such that $\omega - L_0 dt = \mu_i \delta x^i + \omega_0$.

It is easy to see that if the Pfaff form ω is globally or locally singular it is also singular.

A (*global*) *non-Lagrangian system* is given by a Pfaff form ω for which there are two Lagrangians $L, \mu_0 : TM \rightarrow \mathbb{R}$ and a top Pfaff form $\mu = \mu_i dx^i$, such that $\omega - dL = \mu_0 dt + \mu_i dx^i$, thus $\omega_0 = \frac{\partial L}{\partial t} + \mu_0$, $\omega_i = \frac{\partial L}{\partial x^i} + \mu_i$ and $\bar{\omega}_i = \frac{\partial L}{\partial y^i}$. Since $\omega - (\mu_0 + y^i \mu_i) dt = \mu_i \delta x^i + dL$, it follows that a (global) non-Lagrangian system is equivalent to give a globally singular Pfaff form.

We can relax the above condition defining a *local non-Lagrangian system* as a Pfaff form ω such that $\omega - \tilde{\omega} = \mu_0 dt + \mu_i dx^i$, where $\tilde{\omega}$ is a closed form and μ, μ_0 are as previously. In the same way, it follows that a local non-Lagrangian system is equivalent to give a locally singular Pfaff form.

If ω is a local non-Lagrangian system on TM , then it can be proved that it is a global one.

In the case when ω is differentiable only on $TM_* = TM \setminus \{\bar{0}\}$, where $\{\bar{0}\}$ is the image of the null section, then it makes sense to make the difference between a local and a global Pfaff form.

For example, the Pfaff form $\omega = \frac{XY}{\sqrt{(X^2+Y^2)^3}} dx - \frac{X^2}{\sqrt{(X^2+Y^2)^3}} dy$ is a local non-Lagrangian on $\mathbb{R}^2 \times \mathbb{R}_*^2$, not a global one.

Instead of TM_* one can consider another open submanifold of TM .

An other example: the Pfaff form $\omega = Ldt$, associated with a non-constant Lagrangian L , defines a non-Lagrangian system.

Some important class of Pfaff forms are:

– When $\omega_0 = 0$; for example, this is the case of time independent Lagrangians $L = L(x^i, y^i)$, since $\omega_0 = \frac{\partial L}{\partial t}$;

– When $\omega_0 = \omega_i = 0$; for example, this is the case of Lagrangians that depend only on direction: $L = L(y^i)$.

If $\omega = \bar{\omega}_j(y^i) dy^j$, then the equation (8) has the form $\frac{d}{dt} \left(\frac{\partial \bar{\omega}_j}{\partial y^i} \frac{d^2 x_0^j}{dt^2} \right) - \frac{d^2}{dt^2} \bar{\omega}_i = 0$, or $\frac{\partial \bar{\omega}_j}{\partial y^i} \frac{d^2 x_0^j}{dt^2} - \frac{d}{dt} \bar{\omega}_i = c_i \Leftrightarrow \left(\frac{\partial \bar{\omega}_j}{\partial y^i} - \frac{\partial \bar{\omega}_i}{\partial y^j} \right) \frac{d^2 x_0^j}{dt^2} = c_i$.

Example 1. Let us consider coordinates (x, y) on \mathbb{R}^2 and (x, y, X, Y) on $\mathbb{R}^4 = T\mathbb{R}^2$. Let $\omega = YdX - XdY$. The equations (8) have the form: $-\frac{d}{dt} \left(\frac{d^2 y}{dt^2} \right) - \frac{d^2}{dt^2} \left(\frac{dy}{dt} \right) = 0$, or $\frac{d^3 y}{dt^3} = 0$, and $\frac{d}{dt} \left(\frac{d^2 x}{dt^2} \right) + \frac{d^2}{dt^2} \left(\frac{dx}{dt} \right) = 0$, or $\frac{d^3 x}{dt^3} = 0$. The exact solution is: $x(t) = C_1 + C_2 t + C_3 t^2$, $y(t) = C_4 + C_5 t + C_6 t^2$.

Example 2. In \mathbb{R}^2 , as in Example 1. above, let $\omega = -ydx + xdy + YdX - XdY$. The equations (8) have the form $\frac{\partial\omega_j}{\partial x^i} \frac{dx_0^j}{dt} - \frac{d}{dt}(\omega_i + \frac{\partial\bar{\omega}_j}{\partial y^i} \frac{d^2x_0^j}{dt^2}) + \frac{d^2}{dt^2}\bar{\omega}_i = 0$.

For $j = 1$, $\frac{dy}{dt} - \frac{d}{dt}(-y - \frac{d^2y}{dt^2}) + \frac{d^2}{dt^2}(\frac{dy}{dt}) = 0$, or $\frac{dy}{dt} + \frac{d^3y}{dt^3} = 0$ and

For $j = 2$, $-\frac{dx}{dt} - \frac{d}{dt}(x + \frac{d^2x}{dt^2}) - \frac{d^2}{dt^2}(\frac{dx}{dt}) = 0$, or $\frac{dx}{dt} + \frac{d^3x}{dt^3} = 0$.

The general solution is $x(t) = c_1 \cos t + c_3 \sin t + c_5$, $x(t) = c_2 \cos t + c_4 \sin t + c_6$. The integral curves are ellipses and straight lines. If $t_1 < t_2 < t_3$ are given, then for every three distinct points $A_\alpha(x_\alpha, y_\alpha) \in \mathbb{R}^2$, $\alpha = \overline{1, 3}$, there is a unique integral curve in the family that contains the three points, i.e. $t \rightarrow (x(t), y(t))$, $x(t_\alpha) = x_\alpha$, $y(t_\alpha) = y_\alpha$, $\alpha = \overline{1, 3}$.

This feature characterizes the dynamics generated by a third order differential equation, when in general, an integral curve is determined by three distinct points. Let us notice that for a second order differential equation, an integral curve is determined, in general, by two distinct points.

Let us consider now the case $\dim M = 1$. In this case, since the only skew-symmetric matrix of first order is the null matrix, the equation (8) is always of second order, for every Pfaff form $\tilde{\omega} = \omega_0 dt + \omega dx + \bar{\omega} dy$, having the form

$$\left(\frac{\partial^2 \bar{\omega}}{\partial t \partial y} - 2\frac{\partial \omega}{\partial y} + 2\frac{\partial \bar{\omega}}{\partial x} \frac{dx_0}{dt} + \frac{\partial^2 \bar{\omega}}{\partial x^2} \left(\frac{dx_0}{dt}\right)^2\right) + \left(-\frac{\partial^2 \omega}{\partial t \partial y} - \frac{\partial^2 \omega}{\partial x \partial y} - \frac{\partial \omega}{\partial y^2} + 2\frac{\partial^2 \bar{\omega}}{\partial x \partial t} \frac{dx_0}{dt} + \frac{\partial \omega_0}{\partial x} - \frac{\partial^2 \omega_0}{\partial t \partial y} - \frac{\partial \omega}{\partial t} + \frac{\partial^2 \bar{\omega}}{\partial t^2}\right) = 0.$$

In the case when the local functions ω_0 , ω and $\bar{\omega}$ do not depend on y , the above equation becomes

$$2\frac{\partial \bar{\omega}}{\partial x} \frac{d^2 x_0}{dt^2} + \frac{\partial^2 \bar{\omega}}{\partial x^2} \left(\frac{dx_0}{dt}\right)^2 + 2\frac{\partial^2 \bar{\omega}}{\partial x \partial t} \frac{dx_0}{dt} + \frac{\partial \omega_0}{\partial x} - \frac{\partial \omega}{\partial t} + \frac{\partial^2 \bar{\omega}}{\partial t^2} = 0. \quad (17)$$

We can give a global description of this fact. It well-known that any one dimensional manifold is diffeomorphic with \mathbb{R} or S^1 . On \mathbb{R} one can take a single global chart, while on S^1 one can take two charts, where the coordinate functions change by $\frac{\partial x'}{\partial x} = \pm 1$. Using the formula that coordinates change, it follows that if ω_0 , ω and $\bar{\omega}$ do not depend on y on the domains of the two local charts, this is true on the intersection domain; we call a such Pfaff form $\tilde{\omega}$ as a *basic Pfaff form*. We suppose also that $\frac{\partial \bar{\omega}}{\partial x} \neq 0$ in every point, thus $\tilde{\omega}$ is regular

According to [2, Section 2.], a *standard* Lagrangian has the form

$$L(t, x, y) = \frac{1}{2}P(x, t)y^2 + Q(x, t)y + R(x, t). \quad (18)$$

Its Euler-Lagrange equation is $2Px'' + P_x(x')^2 + 2P_t x' + 2(Q_t - R_x) = 0$, where subscripts x , t denote partial derivatives and $x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$. In [2, Proposition 2.1.] one prove that a second order equation

$$x'' + a(t, x)(x')^2 + b(t, x)x' + c(t, x) = 0$$

admits a standard Lagrangian description (18) iff $b_x = 2a_t$; then $P = \exp(2 \int^x a(t, s)ds)$ and $R = \int^x (Q_t(t, s) - c(t, s)P(t, s))ds$, where $Q = Q(x, t)$ is an arbitrary function. The following result can be proved by a straightforward verification.

Proposition 6.2 *The generalized Euler-Lagrange equation of a regular and basic Pfaff form on a one dimensional manifold admits locally standard Lagrangian descriptions.*

Proof. We can prove by a straightforward computation that the equation (17) admits a standard Lagrangian description with

$$a(t, x) = \frac{\partial^2 \bar{\omega}}{\partial x^2}, b(t, x) = \frac{\partial^2 \bar{\omega}}{\partial x \partial t}, c(t, x) = \frac{\frac{\partial \omega_0}{\partial x} - \frac{\partial \omega}{\partial t} + \frac{\partial^2 \bar{\omega}}{\partial t^2}}{2 \frac{\partial \bar{\omega}}{\partial x}}. \square$$

A top Pfaff form $\bar{\alpha}$ and a first order semi-spray $\bar{S} : \mathbb{R} \times TM \rightarrow \mathbb{R} \times T^2M$ having the local forms $\bar{\alpha} = \bar{\alpha}_i(t, x^j, y^j) dx^i$ and $\bar{S}(t, x^j, y^j) = (t, x^i, y^i, \bar{S}^i(t, x^j, y^j))$ give rise to second order Lagrangian L , affine in velocities, given by the formula

$$L(t, x^i, y^i, z^i) = \bar{\alpha}_i(z^i - \bar{S}^i).$$

Let us suppose that there is a map $u : \mathbb{R} \times TM \rightarrow \mathbb{R} \times TM$ of fibered manifolds over $\mathbb{R} \times M$, having the form $u(t, x^i, y^i) = (t, x^i, u^i(t, x^i, y^i))$, such that the semi-spray \bar{S} has the local form $\bar{S}^i(t, x^i, y^i) = u_j^i(t, x^i, y^i) y^j + u^i(t, x^i, y^i)$. Then we can consider the Pfaff form ω given by the formula.

$$\omega = \bar{\alpha}_i dy^i - \bar{\alpha}_j u_j^i dx^i - \bar{\alpha}_j u^j.$$

For example, a 2-form $\alpha \in \mathcal{X}^*(M) \wedge \mathcal{X}^*(M)$, having the local form $\alpha = \frac{1}{2} \alpha_{ij}(x^k) dx^i \wedge dx^j$, gives rise to a top Pfaff form $\bar{\alpha} = \alpha_{ij} y^j dx^i$. Adding a supplementary structure, one can consider a Pfaff form. For example, if ∇ is a linear connection on M , then one can \bar{S} the spray associated with ∇ . Using local coordinates, if $\{\Gamma_{jk}^i\}$ are the local coefficients of ∇ , then $\bar{S}^i = \frac{1}{2} \Gamma_{jk}^i y^j y^k$ are the local coefficients of the first order spray. Then

$$\omega = \alpha_{ij} y^j dy^i - \alpha_{rj} y^j y^k \Gamma_{ik}^r dx^i.$$

A Riemannian metric g on M gives rise to the Levi-Civita connection ∇ . Using local coordinates, if $g = \frac{1}{2} g_{ij}(x^k) dx^i \otimes dx^j$, then $\Gamma_{jk}^i = g^{il} \Gamma_{ijk}$, where (Γ_{ijk}) are the first order Christoffel coefficients $\Gamma_{klj} = \frac{1}{2} \left(\frac{\partial g_{kj}}{\partial x^l} + \frac{\partial g_{lj}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^j} \right)$ and $(g^{ij}) = (g_{ij})^{-1}$. Then $\bar{S}^i(t, x^i, y^i) = g^{ij} \Gamma_{klj} y^k y^l = u_j^i y^j$, $u_j^i = g^{ik} \Gamma_{ilk} y^l$.

The symplectic analogous version can be considered on a *Fedosov manifold*, i.e. a triple (M, α, ∇) , where (M, α) is a symplectic manifold and ∇ is a symplectic linear connection on M , i.e. α is parallel according the ∇ .

Let us consider the canonical symplectic form on \mathbb{R}^{2r} , $\alpha^{(r)} = \varepsilon_{i,i+n} e^i \wedge e^{i+n}$, where $(\varepsilon_{ij}) = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ is the the Levi-Civita tensor on \mathbb{R}^{2r} . Then the second order Lagrangian, affine in accelerations given by $L^{(2)} = \varepsilon_{ij} y^j z^i + k \|y\|^2$, where $\|y\|^2 = \frac{1}{2} \delta_{ij} y^i y^j$. This Lagrangian corresponds to some equivalent Pfaff forms $\omega = \varepsilon_{ij} y^j dy^i + k \delta_{ij} y^j dx^i$ and $\omega' = \varepsilon_{ij} y^j dy^i + k \|y\|^2$ (according to Proposition 3.1). The Pfaff form ω is obtained using the symplectic form (ε_{ij})

and the semi-spray \bar{S} on \mathbb{R}^{2r} having the form $(t, x^i, y^i) \xrightarrow{\bar{S}} (t, x^i, y^i, \bar{S}^i = -y^k \delta_{kj} \varepsilon^{ji})$, where $(\varepsilon^{ij}) = (\varepsilon_{ij})^{-1}$.

The second order Lagrangian, affine in accelerations given by $L^{(2)} = \varepsilon_{ij} y^j z^i + k \|y\|^2$, where $\|y\|^2 = \frac{1}{2} \delta_{ij} y^i y^j$. This Lagrangian corresponds to some equivalent Pfaff forms $\omega = \varepsilon_{ij} y^j dy^i + k \delta_{ij} y^j dx^i$ and $\omega' = \varepsilon_{ij} y^j dy^i + k \|y\|^2$ (according to Proposition 3.1). The Pfaff form ω is obtained using the symplectic form (ε_{ij}) and the semi-spray \bar{S} on \mathbb{R}^{2r} having the form $(t, x^i, y^i) \xrightarrow{\bar{S}} (t, x^i, y^i, \bar{S}^i = -k y^k \delta_{kj} \varepsilon^{ji})$, where $(\varepsilon^{ij}) = (\varepsilon_{ij})^{-1}$.

6.1 Pfaff forms and first order semi-sprays

We show below that for some special cases, the solutions of the generalized Euler-Lagrange equation of a Pfaff form can be given by the integral curves of local first order semi-sprays.

Example 3. Let us consider coordinates (x, y) on \mathbb{R}^2 and (x, y, X, Y) on $\mathbb{R}^4 = T\mathbb{R}^2$. Let $\omega = (X+Y)dX+YdY$. As in Example 1, the equations (8) have the solutions $x'''(t) = y'''(t) = 0$. Using the notations $x = x^1$, $y = x^2$, $X = y^1$, $Y = y^2$, then $\omega = (y^1+y^2)dy^1+y^2dy^2 = \bar{\omega}_1 dy^1 + \bar{\omega}_2 dy^2$, $(h_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(h^{ij}) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The integral solutions of the vector field X are $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = S^i = h^{ij} p_j$, $\frac{dp_i}{dt} = \tilde{\Phi}_i = 0$. It follows that $p_i(t) = p_i^0$, thus $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = c_i$, where $c_1 = -p_1^0$ and $c_2 = p_2^0$. Finally we obtain all the solutions $\frac{d^3 x^i}{dt^2} = 0$. In conclusion, considering arbitrary semi-sprays on \mathbb{R}^2 , with constant coefficients, then we obtain all the solutions of (8) as integral solutions of these first order semi-sprays.

The above example can be extended as follows.

Proposition 6.3 *Let us suppose that there are some coordinates such that the local coefficients of a regular Pfaff form ω depend only on (y^i) . Then there is a family of local semi-sprays of first order whose local coefficients depend only on (y^i) , such that their integral curves project on all the integral curves of the generalized Euler-Lagrange equation of ω .*

Proof. The integral solutions of the vector field X are $\frac{dx^i}{dt} = y^i$, $\frac{dy^i}{dt} = S^i = h^{ij} \left(p_j - \frac{\partial \omega_0}{\partial y^i} - \frac{\partial \omega_j}{\partial y^i} y^j \right)$, $\frac{dp_i}{dt} = \tilde{\Phi}_i = -\frac{\partial \omega_i}{\partial y^j} S^j$. Using the second equation in the expression of the third, it follows that $\frac{dp_i}{dt} = -\frac{\partial \omega_i}{\partial y^j} (y^k) \frac{dy^j}{dt}$, thus $p_i + \omega_i = c_i$ along every solution. It follows that if considering local semi-sprays having as local component functions $\bar{S}^i(y^k) = h^{ij}(y^k) \left(c_j - \omega_j(y^k) - \frac{\partial \omega_0}{\partial y^i}(y^k) - \frac{\partial \omega_j}{\partial y^i}(y^k) y^j \right)$, we obtain all the integral curves of the generalized Euler-Lagrange equation of ω . \square

Since the Pfaff form $\omega' = \omega + dF$ has the same extrema curves as ω , the extrema curves of the Pfaff forms $\omega' = (\omega_0(y^j) + \frac{\partial F}{\partial t}) dt + (\omega_i(y^j) + \frac{\partial F}{\partial x^i}) dx^i +$

$\left(\bar{\omega}_i(y^j) + \frac{\partial F}{\partial y^i}\right) dy^i$ and $\omega = \omega_0(y^j)dt + \omega_i(y^j)dx^i + \bar{\omega}_i(y^j)dy^i$ (used in Proposition above) are the same. In order to detect when one can apply the Proposition above, we prove the following result.

Proposition 6.4 *Let us consider a Pfaff form μ , a point $x_0 \in M$ and a local system of coordinates (U, φ) , where $x_0 \in U$. Then the following statements are equivalent:*

1. *There is a local Pfaff form $\omega = \mu - dF$ on a TU' , $x_0 \in U' \subset U$, such that the local components of ω does depend only on (y^i) .*
2. *The local components of $d\mu$ depend only on (y^i) and the components of $\{dx^i \wedge dt, dx^i \wedge dx^j\}$ vanish.*

Proof. If the property 1. holds for μ , then $d\mu = d\omega$, thus 2. follows. Conversely, let us suppose that 2. holds, thus $d\mu = a_i(y^k)dy^i \wedge dt + b_{ij}(y^k)dx^i \wedge dy^j + \frac{1}{2}c_{ij}(y^k)dy^i \wedge dy^j$. Then we have $0 = dd\mu = \frac{\partial a_i}{\partial y^k}dy^k \wedge dy^i \wedge dt + \frac{\partial b_{ij}}{\partial y^k}dy^k \wedge dx^i \wedge dy^j + \frac{1}{2}\frac{\partial c_{ij}}{\partial y^k}dy^k \wedge dy^i \wedge dy^j$. Thus using the Poincaré Lemma, it follows that $a_i = \frac{\partial f}{\partial y^i}$, $b_{ij} = \frac{\partial g_i}{\partial y^j}$ and $c_{ij} = \frac{\partial h_i}{\partial y^j} - \frac{\partial h_j}{\partial y^i}$ on \mathbb{R}^m , where $f, g_i, h_i : \mathbb{R}^m \rightarrow \mathbb{R}$ are functions that depend only on (y^i) . Let us consider the form $\omega = fdt + g_i dx^i + h_i dy^i$ on $TU = U \times \mathbb{R}^m$. Then $d\mu = d\omega$, or $d(\mu - \omega) = 0$, thus for a sufficiently small $U' \subset U$, $x_0 \in U'$, one have $\mu - \omega = dF$ on TU' . \square

Example 4. Consider the Pfaff form $\omega = -ydx + xdy + YdX - XdY$ on \mathbb{R}^2 used in Example 2., with coordinates (x, y) on \mathbb{R}^2 and (x, y, X, Y) on $\mathbb{R}^4 = T\mathbb{R}^2$. We use below also the notations $x = x^1, y = x^2, X = y^1, Y = y^2$, then $\omega = -x^2 dx^1 + x^1 dx^2 + y^2 dy^1 - y^1 dy^2 = \omega_1 dx^1 + \omega_2 dx^2 + \bar{\omega}_1 dy^1 + \bar{\omega}_2 dy^2$, $(h_{ij}) = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$ and $(h^{ij}) = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$. The integral solutions of the vector field X are $\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} = S^i = h^{ij}p_j, \frac{dp_i}{dt} = \tilde{\Phi}_i = \left(\frac{\partial \omega_i}{\partial x^1} - \frac{\partial \omega_1}{\partial x^i}\right)y^j$. Specifically, $\frac{dp_1}{dt} = 2y^2 = 2\frac{dx^2}{dt}$ and $\frac{dp_2}{dt} = -2y^1 = -2\frac{dx^1}{dt}$. Thus $p_1 = 2x^2 + 2c_1$ and $p_2 = -2x^1 + 2c_2$. Considering the local first order semi-sprays $\bar{S}^1(x^i, y^i) = x^2 + c_1$ and $\bar{S}^2(x^i, y^i) = -x^1 + c_2$, we obtain the system $\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} = \bar{S}^i$. Taking into account the Example 2., the integral curves of all semi-sprays \bar{S} having this form give all the solutions of the generalized Euler-Lagrange equation (8) of ω .

The above example can be extended as follows.

Proposition 6.5 *Let us suppose that there are some coordinates such that the local form of a regular Pfaff form ω is $\omega = \omega_0(y^j) + \omega_i(x^j)dx^i + \bar{\omega}_i(y^j)dy^i$ and $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} = 0$. Then there is a family of local semi-sprays of first order, such that their integral curves project on all the integral curves of the generalized Euler-Lagrange equation of ω .*

Proof. The integral solutions of the vector field X are $\frac{dx^i}{dt} = y^i, \frac{dy^i}{dt} = S^i = h^{ij}\left(p_j - \frac{\partial \omega_0}{\partial y^i}\right), \frac{dp_i}{dt} = \tilde{\Phi}_i = \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j}\right)y^j = -2\frac{\partial \omega_i}{\partial x^j}y^j$. Using Lemma 6.1

below, $\{\omega_i\}$ have the form $\omega_i = c_{ij}x^j + d_i$, thus $p_i = -2c_{ij}x^j + e_j$, where $c_{ij} = -c_{ji}$, d_i and e_i are constants. It follows that considering local semi-sprays having as local component functions $\bar{S}^i(y^k) = h^{ij}(y^k) \left(-2c_{ij}x^j + e_j - \frac{\partial \omega_0}{\partial y^i}(y^k) \right)$, for all constants $\{e_j\}$, we obtain all the integral curves of the generalized Euler-Lagrange equation of ω . \square

Lemma 6.1 *Given the set $\{\omega_i(x^j)\}_{i=\overline{1,m}}$ of real functions on \mathbb{R}^m , then the following conditions are equivalent:*

1. $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} = 0$, $i, j = \overline{1,m}$;
2. there are constants $\{c_{ij}, d_i\}_{i,j=\overline{1,m}}$, $c_{ij} = -c_{ji}$ such that $\omega_i = c_{ij}x^j + d_i$;
3. there is a set $\{\varphi_i\}_{i=\overline{1,m}}$ of real functions on \mathbb{R}^m such that $\frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = \frac{\partial \varphi_i}{\partial x^j}$.

Proof. Obviously 2. implies 1. and 3. Let us suppose that 1. holds. We have $\omega_{ij} = \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} = 2\frac{\partial \omega_i}{\partial x^j}$; then $\frac{\partial \omega_{ij}}{\partial x^k} = 2\frac{\partial^2 \omega_i}{\partial x^j \partial x^k} = \frac{\partial \omega_{ik}}{\partial x^j}$, thus $\frac{\partial^2 \omega_i}{\partial x^j \partial x^k} - \frac{\partial^2 \omega_j}{\partial x^i \partial x^k} = \frac{\partial^2 \omega_i}{\partial x^k \partial x^j} - \frac{\partial^2 \omega_k}{\partial x^i \partial x^j}$, that gives $\frac{\partial \omega_{ik}}{\partial x^j} = 0$, thus 2. holds. Let us suppose that 3. holds. We have $\frac{\partial^2 \omega_i}{\partial x^j \partial x^k} - \frac{\partial^2 \omega_j}{\partial x^i \partial x^k} = \frac{\partial^2 \omega_i}{\partial x^k \partial x^j} - \frac{\partial^2 \omega_k}{\partial x^i \partial x^j}$, thus 3. holds as previously. \square

The Pfaff form $\omega' = \omega + dF$ has the same extrema curves as ω . Thus the extrema curves of the Pfaff forms $\omega' = \frac{\partial F}{\partial t} dt + (\omega_i(x^j) + \frac{\partial F}{\partial x^i}) dx^i + (\bar{\omega}_i(y^j) + \frac{\partial F}{\partial y^i}) dy^i$ and ω from Proposition above are the same. In particular, one can relax the hypothesis of Proposition above, asking the existence of a local function $F(x^i)$ such that $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} = -2\frac{\partial^2 F}{\partial x^i \partial x^j}$; more precisely, $\omega_i + \frac{\partial F}{\partial x^i} = c_{ij}x^j + d_i$, where $c_{ij} = -c_{ji}$ and d_i are constants. In order to apply the result from Proposition above, we prove the following result.

Proposition 6.6 *Let us consider a Pfaff form μ , a point $x_0 \in M$ and a local system of coordinates (U, φ) , where $x_0 \in U$. Then the following statements are equivalent:*

1. There is a local Pfaff form $\omega = \mu - dF$ on a TU' , $x_0 \in U' \subset U$, such that $\omega = \omega_i(x^j)dx^i + \bar{\omega}_i(y^j)dy^i$ and $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \omega_j}{\partial x^i} = 0$.
2. The local components of $d\mu$ have the properties that the components of $\{dx^i \wedge dt, dy^i \wedge dt, dx^i \wedge dy^j\}$ vanish, the components of $\{dy^i \wedge dy^j\}$ depend only on (y^i) and the components μ_{ij} of $\{dx^i \wedge dx^j\}$ are constants.

Proof. If the property 1. holds for μ , then $d\mu = d\omega$, thus 2. follows. Conversely, let us suppose that 2. holds, thus $d\mu = \frac{1}{2}\mu_{ij}dx^i \wedge dx^j + \frac{1}{2}\nu_{ij}(y^k)dy^i \wedge dy^j$, with $\mu_{ij} = -\mu_{ji}$ constants. Using $dd\mu = 0$ and the Poincaré Lemma, it follows that $\nu_{ij} = \frac{\partial g_i}{\partial y^j}(y^k) - \frac{\partial g_j}{\partial y^i}(y^k)$, where $g_i : \mathbb{R}^m \rightarrow \mathbb{R}$. Let us denote $f_i(x^k) = \mu_{ij}x^j$ and consider the local differential form $\omega = f_i(x^k)dx^i + g_i(y^k)dy^i$ on $TU = U \times \mathbb{R}^m$. Then $d\mu = d\omega$, or $d(\mu - \omega) = 0$, thus for a sufficiently small $U' \subset U$, $x_0 \in U'$, one have $\mu - \omega = dF$ on TU' . \square

The Pfaff forms $\omega = \omega_i(x^j)dx^i + \bar{\omega}_i(y^j)dy^i$ and $\omega' = y^i\omega_i(x^j)dt + \bar{\omega}_i(y^j)dy^i$ are equivalent. If $\omega_i(x^j) = c_{ij}x^j$, then $d\omega' = c_{ij}x^jdy^i \wedge dt + c_{ij}y^i dx^j \wedge dt + \frac{\partial \bar{\omega}_i}{\partial y^j}(y^j)dy^i \wedge dy^j$. Then the following result follows in the same line as the previous ones.

Proposition 6.7 *Let us consider a Pfaff form μ , a point $x_0 \in M$ and a local system of coordinates (U, φ) , where $x_0 \in U$. Then the following statements are equivalent:*

1. *There is a local Pfaff form $\omega = \mu - dF$ on a TU' , $x_0 \in U' \subset U$, such that $\omega = \omega_i(x^j)y^i dt + \bar{\omega}_i(y^j)dy^i$ and $\frac{\partial \omega_i}{\partial x^j} + \frac{\partial \bar{\omega}_i}{\partial y^j} = 0$.*
2. *The local components of $d\mu$ have the properties that the components of $\{dx^i \wedge dx^j, dx^i \wedge dy^j\}$ vanish, the components of $\{dy^i \wedge dy^j\}$ depend only on (y^i) and the components f_i of $\{dx^i \wedge dt\}$ and g_j of $\{dy^i \wedge dt\}$ have the property that $\frac{\partial f_j}{\partial y^i} = \frac{\partial g_i}{\partial y^j} = c_{ij}$ are constants.*

7 Appendix

As a manifold, $T^2M \subset TTM$ is the submanifold of the vectors X_v that project according to the double vector bundle structure $\pi^{(2)} : TTM \rightarrow TM$, as tangent bundle of TM and $\pi_*^{(1)} : TTM \rightarrow TM$, as the differential of the canonical projection $\pi^{(1)} : TM \rightarrow M$.

A *slashed* (first order) Lagrangian on M is a differentiable map $L : TM_* \rightarrow \mathbb{R}$, where $TM_* = TM \setminus \{0\}$ and $\{0\}$ is the image of the null section $M \rightarrow TM$. Analogously, a slashed second order Lagrangian on M is a differentiable map $L^{(2)} : T^2M_* \rightarrow \mathbb{R}$, where $T^2M_* = T^2M \setminus \{0\}$ and $\{0\}$ is the image of the „null” section $M \rightarrow T^2M$ given by $(x^i) \rightarrow (x^i, y^i = 0, z^i = 0)$.

Coordinates (x^i) on M , (x^i, y^i) on TM , (x^i, y^i, X^i, Y^i) on T^2M and (x^i, y^i, z^i) on T^2M change according to the rules $x^{i'} = x^{i'}(x^i)$, $y^{i'} = \frac{\partial x^{i'}}{\partial x^i}y^i$, $X^{i'} = \frac{\partial x^{i'}}{\partial x^i}X^i$, $Y^{i'} = y^j \frac{\partial^2 x^{i'}}{\partial x^j \partial x^i}X^i + \frac{\partial x^{i'}}{\partial x^i}Y^i$, $z^{i'} = \frac{1}{2}y^i y^j \frac{\partial^2 x^{i'}}{\partial x^j \partial x^i} + \frac{\partial x^{i'}}{\partial x^i}z^i$ (see, for example, [14]). It follows that some local coordinates (x^i, p_i) on T^*M and (x^i, y^i, p_i, P_i) on T^*TM change according to the rules: $x^{i'}$ and $y^{i'}$ as above, $p_i = p_{i'} \frac{\partial x^{i'}}{\partial x^i}$ and $P_i = \frac{\partial y^{i'}}{\partial x^i}p_{i'} + \frac{\partial x^{i'}}{\partial x^i}P_{i'}$.

On T^2M it can be also considered the coordinates $(x^i, \dot{x}^i, \ddot{x}^i)$, that are more suitable for expressing the derivatives of the functions. The connections between the coordinates $(x^i, \dot{x}^i, \ddot{x}^i)$ and (x^i, y^i, z^i) are $x^i = x^i$, $\dot{x}^i = y^i$, but $\ddot{x}^i = 2z^i$.

There are affine bundles structures $T^2M \rightarrow TM$ and $T^3M \rightarrow T^2M$; in general $T^kM \rightarrow T^{k-1}M$, $k \geq 2$. A (time independent) semi-spray of order k is a section $S : T^kM \rightarrow T^{k+1}M$. Considering the product bundle $\mathbb{R} \times T^{k+1}M \rightarrow \mathbb{R} \times T^kM$, $k \geq 1$, then a (time dependent) semi-spray of order k is a section $S : \mathbb{R} \times T^kM \rightarrow \mathbb{R} \times T^{k+1}M$, such that $S(t, \bar{x}) = (t, \bar{x}, (k+1)S^i(t, \bar{x}))$; this semi-spray of order k is considered in the paper.

The integral curves of a k -order semi-spray S are exactly the integral curves of S regarded as a vector field on T^kM . Using coordinates $(x^i, y^{(1)i}, \dots, y^{(k)i})$

on $T^k M$, the local form of a k -order semi-spray is $S = y^{(1)i} \frac{\partial}{\partial x^i} + 2y^{(2)i} \frac{\partial}{\partial y^{(1)i}} + \dots + ky^{(k)i} \frac{\partial}{\partial y^{(k-1)i}} - (k+1)S^i(x^i, y^{(1)i}, \dots, y^{(k)i})$. We say that S^i are the local functions that give S .

A Pfaff form $\omega \in \mathcal{X}^*(\mathbb{R} \times TM)$ has the local form $\omega = \omega_0(t, x^i, y^i)dt + \omega_i(t, x^j, y^j)dx^i + \bar{\omega}_i(t, x^j, y^j)dy^i$. Then $\omega_0 : \mathbb{R} \times TM \rightarrow \mathbb{R}$ gives a (global defined) real function. The local functions ω_i and $\bar{\omega}_i$ change according to the rules $\bar{\omega}_i = \frac{\partial x^{i'}}{\partial x^i} \bar{\omega}_{i'}$ and $\omega_i = \frac{\partial y^{i'}}{\partial x^i} \bar{\omega}_{i'} + \frac{\partial x^{i'}}{\partial x^i} \omega_{i'}$. We can consider the top components $(\bar{\omega}_i)$ defining a section $\bar{\omega} : \mathbb{R} \times TM \rightarrow \pi^* T^* M$, $\bar{\omega} = \bar{\omega}_i(t, x^j, y^j)dx^i$, of the induced vector bundle $\pi_1 = \pi^*(\pi') : \pi^* T^* M \rightarrow \mathbb{R} \times TM$, where $\pi : \mathbb{R} \times TM \rightarrow M$ comes from the tangent bundle $TM \rightarrow M$ and $\pi' : T^* M \rightarrow M$ is the cotangent bundle of M . We say that $\bar{\omega}$ is a *top Pfaff form* (on M). We can consider a *top differential* d_{top} of the skew symmetric forms generated by the differential algebra generated by top Pfaff forms and the commutative algebra $\mathcal{F}(\mathbb{R} \times TM)$, by relations $d_{top}(f) = \frac{\partial f}{\partial y^i} dx^i$ and $d_{top}(dx^i) = 0$ on the local generators. Then if $\bar{\omega} = \bar{\omega}_i dx^i$ is a top Pfaff form, then $d_{top}(\bar{\omega}) = \left(\frac{\partial \bar{\omega}_i}{\partial y^j} - \frac{\partial \bar{\omega}_j}{\partial y^i} \right) dx^i \wedge dx^j$. A local Poincaré Lemma holds for d_{top} ; if $d_{top}(\bar{\omega}) = 0$, then there is a local Lagrangian $\bar{L} : \mathbb{R} \times TU \rightarrow \mathbb{R}$ such that $\bar{\omega} = d_{top} \bar{L}$. Using a partition of unity on M , subordinated of a local finite open cover $\{U_n\}_{n \in \mathbb{N}}$ with open domains of local charts, one can construct a global Lagrangian $\bar{L} : \mathbb{R} \times TM \rightarrow \mathbb{R}$ such that $\bar{\omega} = d_{top} \bar{L}$.

We say that a Lagrangian $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is *pointed* if $L(t, x^i, y^i = 0) = 0$.

Proposition 7.1 *A Lagrangian $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$ is a pointed one iff there is to top Pfaff form $\nu = \nu_i(t, x^i, y^i)dx^i$, such that $L(t, x^i, y^i) = y^i \nu_i$.*

Proof. We proof the necessity; the sufficiency is obvious. Indeed, if L is pointed, then $L(t, x^i, y^i) = y^i \int_0^1 \frac{\partial L}{\partial y^i}(t, x^i, \tau y^i) d\tau = y^i \nu_i$. It can be easily checked that $\nu = \nu_i dx^i$ is a global top Pfaff form. \square

An other example of a top Pfaff form: if $L^{(2)} : T^2 M \rightarrow \mathbb{R}$ is a second order Lagrangian, affine in accelerations, then $\bar{\omega} = \frac{\partial L^{(2)}}{\partial z^i} dx^i$ is a top Pfaff form. Notice that a top Pfaff form $\bar{\omega} = \bar{\omega}_i(t, x^j, y^j)dx^i$ is a degenerated Pfaff form. Since $\bar{\omega} = \bar{\omega}_i dt + \bar{\omega}_i(dx^i - y^i dt)$, it follows that $\bar{\omega}$ is equivalent to the first order (pointed) Lagrangian $L_0 = \bar{\omega}_i y^i$. Conversely, it is easy to see that a pointed Lagrangian $L_0 = \bar{\omega}_i(t, x^j, y^j)y^i$ is equivalent to the top Pfaff form $\bar{\omega} = \bar{\omega}_i dx^i$.

An analogous object considered in the paper is a *pure Pfaff form* that can be considered as a section $\omega' : \mathbb{R} \times TM \rightarrow \pi_0^* T^* TM$, $\omega' = \bar{\omega}_i(t, x^j, y^j)dy^i + \omega_i(t, x^j, y^j)dx^i$, of the induced vector bundle $\pi_2 = \pi_0^*(\pi'') : \pi_0^* T^* TM \rightarrow \mathbb{R} \times TM$, where $\pi'' : T^* TM \rightarrow TM$ is the cotangent bundle of TM and $\pi_0 : \mathbb{R} \times TM \rightarrow TM$ is the trivial projection.

Let $\pi_E : E \rightarrow M$ and $\pi_F : F \rightarrow M$ be fibered manifolds. The fibered manifold product $P = E \times_M F$ is $P = \bigcup_{x \in M} P_x \subset E \times F$, where $P_x = \{(e, f) \in E \times F : \pi_E(e) = \pi_F(f) = x\}$ is a new fibered manifold $\pi_P : P \rightarrow M$ over M , but also over E and F . The tangent space of P is locally the sum of two subspaces, each tangent to two foliations. Using coordinates, we explicit in

two cases, useful in the paper. First is when $E = TM$ and $F = T^*M$ are the tangent and the cotangent space of M respectively. In this case, considering (x^i) , (x^i, y^i) , (x^i, p_i) , (x^i, y^i, p_i) and $(x^i, y^i, p_i, X^i, Y^i, P_i)$ local coordinates on M , TM , T^*M , $TM \times T^*M$ and $T(TM \times T^*M)$ respectively, then these coordinates change according to the rules $x^{i'} = x^i(x^i)$, $y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$, $p_i = \frac{\partial x^{i'}}{\partial x^i} p_{i'}$, $X^{i'} = \frac{\partial x^{i'}}{\partial x^i} X^i$, $Y^{i'} = \frac{\partial y^{i'}}{\partial x^i} X^i + \frac{\partial x^{i'}}{\partial x^i} Y^i$ and $P_i = \frac{\partial y^{i'}}{\partial x^i} p_{i'} + \frac{\partial x^{i'}}{\partial x^i} P_{i'}$ respectively. A second case is when $E = F = T^*M$ and $T^*M \times_M T^*M = T_2^0 M$. In this case, considering $(x^i, p_{(0)i}, p_{(1)i}, y^i, P_{(0)i}, P_{(1)i})$ local coordinates on $TT_2^0 M$, then $p_{(0)i} = \frac{\partial x^{i'}}{\partial x^i} p_{(0)i'}$, $p_{(0)i} = \frac{\partial x^{i'}}{\partial x^i} p_{(0)i'}$, $y^{i'} = \frac{\partial x^{i'}}{\partial x^i} y^i$, $P_{(0)i} = \frac{\partial y^{i'}}{\partial x^i} p_{(0)i'} + \frac{\partial x^{i'}}{\partial x^i} P_{(0)i'}$, $P_{(1)i} = \frac{\partial y^{i'}}{\partial x^i} p_{(1)i'} + \frac{\partial x^{i'}}{\partial x^i} P_{(1)i'}$.

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